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# UNITS IN SOME FAMILIES OF ALGEBRAIC NUMBER FIELDS

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ABSTRACT. Multi-dimensional continued fractions associated with $GL_n(\mathbf{Z})$ are introduced and applied to find systems of fundamental units in some families of totally real fields and fields with signature (2,1).

## 1. INTRODUCTION

Let $F$ be an algebraic number field of degree $n$. There exist exactly $n$ field embeddings of $F$ in $\mathbf{C}$. Let $s$ be the number of embeddings of $F$ whose images lie in $\mathbf{R}$, and let $2t$ be the number of non-real complex embeddings so that $n = s + 2t$. The pair $(s, t)$ is said to be the *signature* of $F$. Let $\mathbf{Z}_F$ be the ring of integers of the field $F$. A *unit* in $F$ is an invertible element of $\mathbf{Z}_F$. The set of units in $F$ forms a multiplicative group which will be denoted by $\mathbf{Z}_F^\times$. In 1840 P. G. Lejeune-Dirichlet determined the structure of the group $\mathbf{Z}_F^\times$. He showed that $\mathbf{Z}_F^\times$ is a finitely generated Abelian group of rank $r = s + t - 1$, i.e. $\mathbf{Z}_F^\times$ is isomorphic to $\mu_F \times \mathbf{Z}^r$, where $\mu_F$ is a finite cyclic group. $\mu_F$ is called the *torsion* subgroup of $\mathbf{Z}_F^\times$. Thus, there exist units $\epsilon_1$, ..., $\epsilon_r$ such that every element of $\mathbf{Z}_F^\times$ can be written in a unique way as $\zeta\epsilon_1^{n_1}...\epsilon_r^{n_r}$, where $n_i \in \mathbf{Z}$ and $\zeta$ is a root of unity in $F$. Such a set $\{\epsilon_1, ..., \epsilon_r\}$ is called a *system of fundamental units* of $F$. Finding a system of fundamental units of $F$ is one of the main computational problem of algebraic number theory (see e.g. [6], p. 217). Much work has been done to solve this problem for certain classes of algebraic number fields (see e.g. [22]). In the case of the real quadratic fields, the continued fraction algorithm provides a very efficient method for solving this problem (see e.g. [22], p. 119). This approach goes back to L. Euler, who applied continued fractions to solve Pell's equation $x^2 - dy^2 = \pm 1$. (If a square-free positive integer $d \equiv 2$ or $3 \bmod 4$ and $x, y$ is an integral solution of this equation, then $x + \sqrt{d}y$ is a unit in the real quadratic field $\mathbf{Q}(\sqrt{d})$. Moreover, any unit in $\mathbf{Q}(\sqrt{d})$ can be obtained this way.) Many attempts have been made to develop a similar algorithm that would find a system of fundamental units in other algebraic number fields. In the case of a cubic field, one of the most successful such algorithms was introduced by G. F. Voronoi [28]. A review of the multi-dimensional continued fraction algorithms and their properties that were known by 1980 can be found in [2].

In [30] and [31], a continued fraction algorithm associated with a discrete group acting in a hyperbolic space was defined. The purpose of the present paper is to extend this definition to the case of the group $\Gamma = \mathrm{GL}_n(\mathbf{Z})/\{\pm 1\}$ acting on

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$\mathcal{P} = SL_n(\mathbf{R})/SO_n(\mathbf{R})$ and apply it to the problem of finding a system of fundamental units in an algebraic number field $F$. The symmetric space $\mathcal{P}$ can be identified with the set of definite quadratic forms in $n$ real variables with the leading coefficient one. $\mathcal{P}$ with the metric $ds^2 = \operatorname{trace}((X^{-1}dX)^2)$, where $X = (x_{ij}) \in \mathcal{P}$ and $dX = (dx_{ij})$, is a Riemannian manifold (see e.g. [13] or [27]).

Assume that $g \in GL_n(\mathbf{R})$. Let $ga_i = \lambda_i a_i$, $i = 1, \ldots, n$, so that $a_i$ is an eigenvector of $g$ corresponding to its eigenvalue $\lambda_i$. For simplicity, assume that all the eigenvalues of $g$ are distinct. Let $P = (a_1, \ldots, a_n)$ be the matrix with columns $a_1, \ldots, a_n$. The set of points in $\mathcal{P}$ fixed by $g$ will be called the *axis* $L_P$ of $g$. The axis $L_P$ of $g$ depends only on eigenvectors of $g$, i.e. on $P$, but not on its eigenvalues (see Section 3). $L_P$ is a totally geodesic submanifold of $\mathcal{P}$ of dimension $s + t - 1$, where $s$ is the number of real and $2t$ the number of non-real complex eigenvalues of $g$, so that $n = s + 2t$. If $t = 0$, then $L_P$ is an $(n-1)$-flat in $\mathcal{P}$ (see e.g. [13]).

Let $(1, \omega_2, \ldots, \omega_n)$ be a $\mathbf{Z}$-basis of the ring of integers $\mathbf{Z}_F$ of a number field $F$ of degree $n$. Let $a_1 = (1, \omega_2, \ldots, \omega_n)^T$. Let $\gamma \in \mathbf{Z}_F$. Then $\gamma\omega_i = \sum m_{ij}\omega_j$ or $\gamma a_1 = M_\gamma a_1$, where $\omega_1 = 1$, $m_{ij} \in \mathbf{Z}$ and $M_\gamma = (m_{ij})$ is a square matrix of order $n$. Let $\sigma_i$ be the $n$ distinct embeddings of $F$ in $\mathbf{C}$. Let $a_i = \sigma_i(a_1)$ and $\gamma_i = \sigma_i(\gamma)$, where $\gamma_1 = \gamma$. Then $\gamma_i a_i = M_\gamma a_i$ for $i = 1, \ldots, n$. Thus, $a_i$ is an eigenvector of $M_\gamma$ corresponding to its eigenvalue $\gamma_i$. It is clear that the map $\gamma \longmapsto M_\gamma$ is an isomorphism of the ring of integers $\mathbf{Z}_F$ and the commutative ring of $\mathbf{Z}$-integral square matrices of order $n$ with the common axis $L_P$. The norm of $\gamma$ equals $\det(M_\gamma)$, so that $\gamma$ is a unit in $\mathbf{Z}_F$ if and only if $M_\gamma \in \mathrm{GL}_n(\mathbf{Z})$. The torsion-free subgroup $\Gamma_L$ of the stabilizer of $L_P$ is isomorphic to $\mathbf{Z}_F^\times/\mu_F$. Thus, the problem of finding a system of fundamental units of $F$ is equivalent to the problem of finding a set of generators of $\Gamma_L$. The [multi-dimensional continued fraction] Algorithm II introduced in this paper can be used to solve the latter problem. Here, a set of generators of $\Gamma_L$ and therefore a system of fundamental units is found in some families of fields $F$ of degree $n \leq 4$.

In Section 2, the notion of the height of a point in $\mathcal{P}$ is introduced. Let $w = (1, 0, \ldots, 0)^T$ and $W = ww^T$. In what follows, the point $W$, which belongs to the boundary of $\mathcal{P}$, is analogous to the point $\infty$ in the upper half-space model $H^{n+1} = \{(z, t) : z \in \mathbf{R}^n, t > 0\}$ of the $(n+1)$-dimensional hyperbolic space (see [30] and [31]). The set $K = K(w)$ in $\mathcal{P}$ is defined so that, for every point $X \in \mathcal{P}$, the points in the $\Gamma$-orbit of $X$ with the largest height belong to $K(w)$. The images $K[g]$ of $K$, $g \in \Gamma$, under the action of $\Gamma$ form the *K-tessellation* of $\mathcal{P}$. The $K$-tessellation of $\mathcal{P}$ is $\Gamma$-invariant.

If $L_P \cap K[g] \neq \varnothing$, $g \in \Gamma$, then the vector $u = g^{-1}w \in \mathbf{Z}^n$ is called a *convergent* of $L_P$. In Section 3, it is shown that if $u$ is a convergent of $L_P$, then $|\langle a_1, u\rangle \ldots \langle a_n, u\rangle / \det P|$, where $\langle , \rangle$ denotes the dot product in $\mathbf{R}^n$, is small (Theorem 7). Algorithm II, which is introduced in Section 3, can be applied to find the sets $R(g^{-1}w) = L_P \cap K[g] \neq \varnothing$, which form a tessellation of $L_P$, and the set of convergents of $L_P$.

If $g \in \Gamma$, then there are only finitely many sets $R(u)$ which are not congruent modulo the action of $\Gamma$. The union of non-congruent sets $R(u)$ forms a fundamental domain of $\Gamma_L$ in $L_P$. Assume that the characteristic polynomial $p(x)$ of $g$ is irreducible. Let $p(\epsilon) = 0$. In Section 4, the problem of finding a system of fundamental units in $F$ is solved for some families of totally real fields $F = \mathbf{Q}(\epsilon)$ by reducing it, as explained above, to the problem of finding a set of generators of $\Gamma_L$. In Examples

1-4, new proofs of certain known results are given. The new results obtained in Examples 5 and 6 can be presented as follows.

**Theorem 1.** *Assume that $t > 3$ is a positive integer. Let $f(x) = x^4 + tx^3 - x^2 - tx + \alpha = x(x^2 - 1)(x + t) + \alpha$, $\alpha = \pm 1$. Let $f(\epsilon) = 0$. Assume that $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a $\mathbf{Z}$ -basis of the maximal order $\mathbf{Z}_F$ of the totally real quadric field $F = \mathbf{Q}(\epsilon)$. Then $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon - 1, \epsilon, \epsilon + 1\rangle$.*

Note that the Galois group of $F$ is $D_4$ if $\alpha = 1$, and it is $S_4$ if $\alpha = -1$.

In Section 5, the problem of finding a system of fundamental units is solved for some families of fields with signature $(2, 1)$. The following theorems are proved in Examples 7, 8, and 9 respectively.

**Theorem 2.** *Let $f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2 + 1)(x + t) + 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that $\eta = \epsilon + \epsilon^{-1} = (-t \pm \sqrt{t^2 + 4})/2$ is a fundamental unit of the quadratic subfield $K = \mathbf{Q}(\sqrt{d})$, $d = t^2 + 4$, of the dihedral quartic field $F = \mathbf{Q}(\epsilon)$ with signature $(2, 1)$. Assume that $4t^2-9$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a $\mathbf{Z}$-basis of the maximal order $\mathbf{Z}_F$ of the field $F$, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \eta\rangle$.*

**Theorem 3** (cf. [20])**.** *Let $t \geq 4$ be an integer. Let $f(x) = x^4 + tx^3 + x^2 + tx - 1 = x(x^2 + 1)(x + t) - 1$. Let $\epsilon$ be a real root of $f(x)$. Assume that the discriminant of the quartic field $F = \mathbf{Q}(\epsilon)$ with signature $(2, 1)$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a $\mathbf{Z}$-basis of the maximal order $\mathbf{Z}_F$ of the field $F$, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon + t\rangle$.*

**Theorem 4.** *Let $\alpha = \pm 1$. Let $t \geq s$ be positive integers and let $\eta = \frac{1}{2}(t+\sqrt{d})$, where $d = t^2 + 4\alpha$. Let $f(x) = f_1(x)f_2(x) = x^4 + stx^3 + (t - \alpha s^2)x^2 + s(t^2 + 2\alpha)x - \alpha$, where $f_1(x) = (x^2 + s\eta x - 1/\eta)$, $f_2(x) = (x^2 - sx/\eta + \eta)$. Let $f_1(\epsilon) = 0$. Then $\eta \in F = \mathbf{Q}(\epsilon)$. Assume that $\eta$ is a fundamental unit of the quadratic subfield $K = \mathbf{Q}(\sqrt{d})$ of $F$ and that $\Delta = 4s^2t^3 + 12\alpha s^2 t - s^4 + 16\alpha$ is square-free. Denote $p(x) = (s + (\alpha t - s^2)x + \alpha x^3)/(ts^2 + 1)$. Then $\{1, \epsilon, \epsilon^2, p(\epsilon)\}$ is a $\mathbf{Z}$-basis of the maximal order $\mathbf{Z}_F$ of the field $F$, the discriminant of $F$ is $d^2\Delta$, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \eta\rangle$.*

Families of cyclic quartic fields are considered in [15] and [35]. In [33], a one-dimensional version of Algorithm II is applied to find fundamental units in a two-parameter family of complex cubic fields.

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## 2. Fundamental domains and $K$-tessellation

Let $n \geq 2$ be a positive integer. Let $V_n$ be the space of symmetric $n \times n$ real matrices. The dimension of $V_n$ is $N = n(n+1)/2$. The action of $g \in G = GL(n, \mathbf{R})$ on $X \in V_n$ is given by

$$X \longmapsto X[g] = g^T X g.$$

For a subset $S$ of $V_n$, denote $S[g] = \{X[g] \in V_n : X \in S\}$.

The one-dimensional subspaces of $V_n$ form the real projective space $V$ of dimension $N - 1$, so that for any fixed nonzero $X \in V_n$, all the vectors $kX \in V_n$, $0 \neq k \in \mathbf{R}$, represent one point in $V$. Denote by $\mathcal{P} \subset V$ the set of (positive) definite elements of $V$ and by $\mathcal{B}$ the boundary of $\mathcal{P}$ ($\mathcal{B}$ can be identified with the non-negative elements of $V$ of rank less than $n$). The group $G$ preserves both $\mathcal{P}$ and $\mathcal{B}$, as does its arithmetic subgroup $GL(n, \mathbf{Z})$.

The space $V_n$ (and $V$) can be also identified with the set of quadratic forms $A[x] = x^T Ax$, $A \in V_n$, $x \in \mathbf{R}^n$. With each point $a = (a_1, ..., a_n)^T \in \mathbf{R}^n$, we associate the matrix $A = aa^T \in \mathcal{B}$ and the quadratic form

$$A[x] = \langle a, x\rangle^2 = (a_1x_1 + ... + a_nx_n)^2 \tag{1}$$

of rank one. For $g \in G$, we have $\langle ga, x\rangle = a^T g^T x = \left\langle a, g^T x\right\rangle$.

Let $w = (1, 0, ..., 0)^T$ and $W = ww^T$. Then $\langle w, x\rangle^2 = x_1^2$ and $W[g] = U = uu^T$, where $u = g^T w$.

Denote by $G_\infty$ and $\Gamma_\infty$ the stabilizers of $w$ in $G$ and $\Gamma = GL(n, \mathbf{Z})/\{\pm 1\}$ respectively. Then

$$G_\infty = \{g \in G : gw = w\} = \{g \in G : g_1 = w\},$$

where $g_1$ is the first column of $g$. Thus, $g \in G_\infty$ if and only if $W[g^T] = W$.

We shall say that $A \in V$ is *extremal* if $|A[x]| \geq |A[w]| = a_{11}^2$ for any $x \in \mathbf{Z}^n$, $x \neq (0, ..., 0)$. Let $\mathcal{A}_n = \{X \in V : X[w] \neq 0\}$. It is clear that $\mathcal{P} \subset \mathcal{A}_n$. For $X \in \mathcal{A}_n$, we shall say that

$$\mathrm{ht}(X) = |\det(X)|^{1/n}/|X[w]|$$

is the *height* of $X$ and, for a subset $S$ of $V$, we define the *height* of $S$ as

$$\mathrm{ht}(S) = \max \mathrm{ht}(X), \quad X \in S.$$

The elements of $\mathcal{A}_n$ will be normalized so that $X[w] = 1$. For a fixed $g \in \Gamma$, the set $\{X \in \mathcal{A}_n : |X[gw]| < 1\}$ is called the *g-strip* (cf. [32], [29], where this definition is introduced for $n = 2$). It is clear that the $gh$-strip coincides with the $g$-strip for any $h \in \Gamma_\infty$. The plane

$$L^+(gw) = L^+(g) = \{X \in \mathcal{A}_n : X[gw] = 1\}$$

is the boundary of the $g$-strip which cuts $\mathcal{P}$. Let $\mathcal{R}_w$ be the set of all extremal points of $V$. Denote

$$K = K(w) = \mathcal{P} \cap \mathcal{R}_w.$$

(In the notation of [1], p.148, $K$ is the dual core of $K_{\mathrm{perf}}$.) Note that $K \subset \mathcal{A}_n$ is bounded by the planes $L^+(g)$. If $h \in \Gamma_\infty$, then $X[hw] = X[w]$ and, therefore, $\mathrm{ht}(X[h]) = \mathrm{ht}(X)$. Thus,

$$K[h] = K, \qquad h \in \Gamma_\infty. \tag{2}$$

Let $q(x)$ be an indefinite quadratic form in $n > 2$ variables. By Margulis' theorem [19], if $q(x)$ is not a multiple of a quadratic form with integer coefficients, then the infimum of $|q(x)|$ taken over all $x \in \mathbf{Z}^n$, $x \neq (0, ..., 0)$, is equal to zero. Hence, all the points of $\mathcal{R}_w - \mathcal{P}$ are rational if $n > 2$. By Meyer's theorem (see e.g. [5]), if the coefficients of $q(x)$ are rational and $n = 5$, then there is $x_0 \in \mathbf{Z}^5$, $x_0 \neq (0, ..., 0)$, such that $q(x_0) = 0$. Thus, $\mathcal{R}_w - \mathcal{P}$ is empty and $K = \mathcal{R}_w$ if $n > 4$.

Let $D$ be any of the fundamental domains of $\Gamma$ obtained by Minkowski, Korkine and Zolotarev (see e.g. [24], p.13), or Grenier [16]. For $X \in D$ we have $X[w] = \inf X[gw]$, $g \in \Gamma$, in any of these cases. Hence, $\bigcup D[g] = K$, the union being taken over all $g \in \Gamma_\infty$. Note that the fundamental domain described in [16] coincides with the domain found by Korkine and Zolotarev in 1873 (see [18] or [24]). Unless a point $X \in \mathcal{P}$ is integral, like point $I$ in Example 1 below, in order to prove that $X$ is extremal, we shall show that $X[h]$ is Minkowski reduced for some $h \in \Gamma_\infty$.

The main features of our approach to the problem of finding a system of fundamental units in an algebraic number field can be seen in the following simple example.

**Example 1.** Let $\Gamma = GL_2(\mathbf{Z})$. Let $X = (x_{ij}) \in V$. A point $X$ lies in $\mathcal{B}$ if and only if $\det(X) = x_{11}x_{22} - x_{12}^2 = 0$. Thus, when $n = 2$, $\mathcal{B}$ is a conic in the projective plane $V$, and $\mathcal{P}$, which consists of the points $X$ with $\det(X) > 0$, is the Klein model of the hyperbolic plane. Let $f(x) = x^2 - tx - 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that either $t^2 + 4$ or $t^2/4 + 1$ is a square-free integer. Then $\{1, \epsilon\}$ is a $\mathbf{Z}$-basis of the maximal order $\mathbf{Z}_F$ of the field $F = \mathbf{Q}(\epsilon)$. Let

$$E = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}.$$

Then $f(x)$ is a characteristic polynomial of $E$; and $a_1 = (1, \epsilon)^T$ and $a_2 = (1, -1/\epsilon)^T$ are eigenvectors of $E$ corresponding to its eigenvalues $\epsilon$ and $-1/\epsilon$ respectively. Let $A_i = a_i a_i^T$. Let $L_P$ be the axis of $E$. Then $L_P$ is the interval $\mu A_1 + (1-\mu)A_2$, $0 < \mu < 1$, which is a geodesic in $\mathcal{P}$. The identity matrix $I$ is the intersection of $L_P$ with $L^+(E)$, and the interval $(I, I[E])$ is a fundamental domain of $\Gamma_L$ on $L_P$. Thus, $\Gamma_L = \langle E \rangle$ and, therefore, $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon \rangle$.

The period length of the corresponding continued fraction is one (see Remark 2 at the end of Section 3). More examples of this type can be found in [30] and [31].

Similar families of algebraic number fields of degree three and four are considered in Sections 4 and 5.

By (2), $K[hg] = K[g]$ for any $g \in \Gamma$ and $h \in \Gamma_\infty$. Thus, the sets $K[g]$ are parameterized by the classes $\Gamma_\infty \backslash \Gamma$ or by primitive vectors $u = g^{-1}h^{-1}w = g^{-1}w$, so that $\pm u$ represent the same $K[g]$. The sets $K[g]$, $g \in \Gamma_\infty \backslash \Gamma$ , form a tessellation of $\mathcal{P}$ which will be called the *$K$-tessellation.* It is clear that the $K$-tessellation of $\mathcal{P}$ is $\Gamma$-invariant. The vertices of $K$ are called the *perfect forms* (see e.g. [1] or [24]). The perfect forms are known for $n \le 7$.

## 3. Axes of elements of $G$

Let $g \in G$. Let $g a_i = \lambda_i a_i$, $i = 1, ..., n$, where, for simplicity, we assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Here $a_i$ is an eigenvector of $g$ corresponding to its eigenvalue $\lambda_i$. Assume that $\langle a_i, w \rangle \neq 0$, $i = 1, ..., n$. Then we can choose $a_i$ so that

$$\langle a_i, w \rangle = 1, \quad i = 1, ..., n. \tag{3}$$

$g \in \Gamma$ is said to be *irreducible* if its characteristic polynomial is irreducible over $\mathbf{Q}$. If $g \in \Gamma$ is irreducible, then all its eigenvalues are distinct. Assume that $\lambda_1, \lambda_2, ..., \lambda_s$ are real and $\lambda_{s+1}, \lambda_{s+2}, ..., \lambda_{s+t}, \overline{\lambda}_{s+1}, \overline{\lambda}_{s+2}, ..., \overline{\lambda}_{s+t}$ are non-real complex eigenvalues of $g$, so that $n = s + 2t$. Let $\lambda_k \neq \pm 1$. Let

$$P = (a_1, ..., a_s, a_{s+1}, \overline{a}_{s+1}, ..., a_{s+t}, \overline{a}_{s+t})$$

be the matrix with columns $a_1, ..., a_s, a_{s+1}, \overline{a}_{s+1}, ..., a_{s+t}, \overline{a}_{s+t}$ and let

$$H = \operatorname{diag}(\lambda_1, ..., \lambda_s, \lambda_{s+1}, \overline{\lambda}_{s+1}, ..., \lambda_{s+t}, \overline{\lambda}_{s+t}).$$

Then $g = PHP^{-1}$. For $k > s$, let $a_k = \alpha_k + i\beta_k$, where $\alpha_k, \beta_k \in \mathbf{R}^n$. Then

$$\det P = (2i)^t \det(a_1, ..., a_s, \beta_{s+1}, \alpha_{s+1}, ..., \beta_{s+t}, \alpha_{s+t}).$$

The totally geodesic submanifold $L_P$ of $\mathcal{P}$ fixed by $g = PHP^{-1}$ will be called the *axis* of $g$. The dimension of $L_P$ is $s+t-1$. It can be identified with the set of quadratic forms in $\mathcal{A}_n$

$$q[x] = \sum_{i=1}^{s} \mu_i \langle x, a_i\rangle^2 + \sum_{i=1}^{t} \mu_{s+i} |\langle x, a_{s+i}\rangle|^2, \quad \mu_i \geq 0,\ \sum_{i=1}^{s+t} \mu_i = 1. \tag{4}$$

or

$$q[x] = \sum_{i=1}^{s} \mu_i \langle x, a_i\rangle^2 + \sum_{i=1}^{t} \mu_{s+i}(\langle x, \alpha_{s+i}\rangle^2 + \langle x, \beta_{s+i}\rangle^2).$$

Hence

$$\det q = (-4)^{-t} \mu_1 ... \mu_s \mu_{s+1}^2 ... \mu_{s+t}^2 (\det P)^2.$$

It follows from (4) that $L_P$ is the axis of $h \in G$ if and only if $a_i$, $i = 1, ..., n$, are eigenvectors of $h$. Hence, the axis of $g$ depends only on its set of eigenvectors, i.e. on $P$, but not on the eigenvalues of $g$. A point $q \in L_P$ can be also represented as

$$q = \sum_{i=1}^{s+t} \mu_i A_i, \quad A_i = a_i a_i^T,\ i \leq s, \quad A_{s+i} = \alpha_{s+i}\alpha_{s+i}^T + \beta_{s+i}\beta_{s+i}^T.$$

Thus, $L_P$ is the simplex with vertices $A_i$, $i = 1, ..., s+t$. All the faces of $L_P$ belong to $\mathcal{B}$. Note that $L_P[g^T] = L_P$. The curve $q[g^v] = (\sum \mu_i |\lambda_i|^{2v} A_i)/\sum \mu_i |\lambda_i|^{2v}$ is a geodesic in $L_P$ through $q$. If $|\lambda_1| < |\lambda_2| < ... < |\lambda_{s+t}|$, then $q[g^v] \to A_1$ as the real parameter $v \to -\infty$, and $q[g^v] \to A_{s+t}$ as $v \to \infty$. Note that if $t = 0$, then $L_P$ is an $(n-1)$-flat in $\mathcal{P}$ and the stabilizer of $L_P$ in $G$ is a maximal commutative subgroup of $G$ (see e.g. [13]).

Denote $K(g^{-1}w) = K[g]$ and

$$R(g^{-1}w) = K[g] \cap L_P \neq \varnothing, \qquad g \in \Gamma_\infty \backslash \Gamma. \tag{5}$$

The sets $R(u)$, $u = g^{-1}w$, form a tessellation of $L_P$, which is invariant modulo the action of $\Gamma$ since the $K$-tessellation of $\mathcal{P}$ is $\Gamma$-invariant. We say that this tessellation is *periodic* if there are only a finite number of non-congruent sets $R(u)$ modulo the action of $\mathrm{Stab}(L_P, \Gamma)$. In that case, the union of all non-congruent sets $R(u)$ is a fundamental domain of $\mathrm{Stab}(L_P, \Gamma)$. The number of non-congruent sets $R(u)$ in the tessellation of $L_P$ will be called the *period length.*

Denote

$$\gamma_n = 1/h_n, \quad h_n = \inf\ \mathrm{ht}(A), \quad A \in K(w).$$

Hermite's constant $\gamma_n$ is known for $n \leq 8$ (see e.g. [4]). Let $C_n = \gamma_n/n$. Then

$$C_2 = 1/\sqrt{3}, \quad C_3 = 2^{1/3}/3, \quad C_4 = 1/\sqrt{8}, \quad C_5 = 8^{1/5}/5,$$

$$C_6 = 1/3^{7/6}, \quad C_7 = 64^{1/7}/7, \quad C_8 = 1/4,$$

and for large $n$ (see [8]),

$$\frac{1}{2\pi e} \leq C_n \leq \frac{1.744}{2\pi e}.$$

Let

$$N_P(x) = \langle x, a_1\rangle ... \langle x, a_n\rangle = \langle x, a_1\rangle ... \langle x, a_s\rangle\, |\langle x, a_{s+1}\rangle ... \langle x, a_{s+t}\rangle|^2,$$

where $\langle x, a_i\rangle = x^T a_i$. Define

$$\nu(L_P) = \inf \left| \frac{N_P(gw)}{\det P} \right|,$$

where the infimum is taken over all $g \in \Gamma$. It is clear that $\nu(L_P) = \nu(L_{MP}[h])$ for any $h \in \Gamma$, and $M = \mathrm{diag}(\mu_1, ..., \mu_s, \mu_{s+1}, \overline{\mu}_{s+1}, ..., \mu_{s+t}, \overline{\mu}_{s+t})$, where $\mu_1, ..., \mu_s \in \mathbf{R}$, $\mu_{s+1}, ..., \mu_{s+t} \in \mathbf{C}$ and $\mu_1 ... \mu_{s+t} \neq 0$. The projective invariant $\nu(L_P)$ is well known in the geometry of numbers (see e.g. [4] or [17]). In particular, for $n = s = 3$, it was shown by Davenport [9], [10], [11] that $\nu(L_P) \leq 1/7$, where the equality holds only if $ga_i = (1, \alpha_i, \alpha_i^2)$, $i = 1, 2, 3$, the $\alpha_i$ being the roots of $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$, for some $g \in \Gamma$. For other matrices $P$, $\nu(L_P) \leq 1/9$. Swinnerton-Dyer [26] found the consequent seventeen values of $\nu(L_P)$.

A point $q_m \in L_P$ is said to be the *summit* of $L_P$ if $|\det(q_m)| = \max |\det(q)|$, the maximum being taken over all $q \in L_P$. It is clear that if $R = L_P \cap K(w) \neq \varnothing$, then $q_m \in R$.

**Lemma 5.** *Let $L_P$ be the totally geodesic manifold fixed by $g \in G$ and defined by (4), where $ga_i = \lambda_i a_i$. Let $P = (a_1, ..., a_n)$ be the matrix with columns $a_1, ..., a_n$. Then*

$$q_m[x] = \frac{1}{n} \sum_{i=1}^{s} \langle x, a_i \rangle^2 + \frac{2}{n} \sum_{i=1}^{t} |\langle x, a_{s+i} \rangle|^2 \tag{6}$$

*is the summit of $L_P$,*

$$\mathrm{ht}(L_P) = \frac{1}{n} \left| \frac{\det P}{N_P(w)} \right|^{2/n},$$

*and*

$$\nu(L_P) = \inf(n\mathrm{ht}(L_P[g]))^{-n/2}, \ g \in \Gamma.$$

*Proof.* The maximum of $m = \mu_1 ... \mu_s \mu_{s+1}^2 ... \mu_{s+t}^2$ subject to $\mu_i \geq 0$, $\sum_{i=1}^{s+t} \mu_i = 1$, is attained when $\mu_k = 1/n$, $k \leq s$, and $\mu_k = 2/n$, $k > s$. Thus, $\max m = 4^t n^{-n}$, (6) holds, and $\max \det(q) = \det(q_m) = |\det P|^2 n^{-n}$. By ( 3), $N_P(w) = q[w] = 1$. Hence

$$\mathrm{ht}(L_P) = |\det(q_m)|^{1/n} = \frac{1}{n} \left| \frac{\det P}{N_P(w)} \right|^{2/n},$$

and $\nu(L_P) = \inf(n\mathrm{ht}(gL_P))^{-n/2}$, $g \in \Gamma$, as required. □

It follows that, for any $L_P$, $\mathrm{ht}(X) = |\det(X)|^{1/n} \to 0$ as $X \in L_P$ approaches the boundary of $L_P$. Note that $nq_m[x]$ is the form size$(M_x)$ from [6], p. 169.

In Example 1 above, by Lemma 5, $q_m = (A_1 + A_2)/2 = (I + I[E])/2$ and $\det(q_m) = t^2/4 + 1$ and $\mathrm{ht}(L_P) = \sqrt{t^2/4 + 1}$.

Assume that $L_P \cap K(w) = \varnothing$. Let $q_m$ be the summit of $L_P$. Since $q_m \notin K(w)$, there is $g \in \Gamma$ such that $\mathrm{ht}(L_P[g]) \geq \mathrm{ht}(q_m[g]) > \mathrm{ht}(q_m) = \mathrm{ht}(L_P)$. We have obtained the following.

**Lemma 6.** *Let $L_P$ be the totally geodesic manifold fixed by $g \in G$ and defined by (4), where $ga_i = \lambda_i a_i$. Then*

$$\nu(L_P) = \inf(n\mathrm{ht}(L_P[g_j]))^{-n/2}, \quad L_P \cap K(g_j w) \neq \varnothing, \quad g_j \in \Gamma.$$

By Lemma 6,

$$\nu(L_P) < (nh_n)^{-n/2} = C_n^{n/2}.$$

For $n = 2$, $\nu(L_P) < C_2 = 1/\sqrt{3}$. $\sup \nu(L_P) = 1/\sqrt{5}$ is a well-known approximation constant. It is attained only if $ga_i = (1, (1 \pm \sqrt{5})/2)$ for some $g \in \Gamma$. For other matrices $P$, $\nu(L_P) \leq 1/\sqrt{8}$ (cf. Example 1 above, where, for $t = 1$ and 2, the discriminants of the fields $F$ are 5 and 8). The set of values of $\nu(L_P) > 1/3$ is discrete and consists of numbers $(9-4/m^2)^{-1/2}$, where $m$ runs through the set of all positive integers such that $(m, m_1, m_2)$ is a solution of the Diophantine equation $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$. This result was obtained by Markov in 1879 (see e.g. [17], Section. 43).

When $n = 3$, we have $\nu(L_P) < \sqrt{2/27} = 0.2722$. This estimate does not depend on the dimension of $L_P$. It was shown by Davenport [12] that $\nu(L_P) \leq 1/\sqrt{23} = 0.2085$ when $s = t = 1$, where the equality holds only if $ga_i = (1, \alpha_i, \alpha_i^2)$, $i = 1, 2, 3$, the $\alpha_i$ being the roots of $\alpha^3 - \alpha - 1 = 0$, for some $g \in \Gamma$. As mentioned above, $\nu(L_P) \leq 1/7 = 0.1429$ when $\dim L_P = 2$.

Assume that $L_P \cap K(gw) \neq \varnothing$, where $g \in \Gamma$. Since $L_P[g] \cap K(w) \neq \varnothing$, by Lemma 5,

$$\mathrm{ht}(L_P[g]) = \mathrm{ht}(L_{g^TP}) = \frac{1}{n}\left|\frac{\det P}{N_{g^TP}(w)}\right|^{2/n} > h_n = 1/\gamma_n.$$

But $N_{g^TP}(x) = \left\langle x, g^Ta_1\right\rangle ... \left\langle x, g^Ta_n\right\rangle = \left\langle gx, a_1\right\rangle ... \left\langle gx, a_n\right\rangle$. Hence $N_{g^TP}(w) = N_P(gw)$.

A vector $gw \in \mathbf{Z}^n$ such that $L_P \cap K(gw) \neq \varnothing$ will be called a *convergent* of $L_P$. We have proved the following.

**Theorem 7.** *If a vector $u$ is a convergent of $L_P$ (that is, if $L_P \cap K(u) \neq \varnothing$), then*

$$|N_P(u)| < C_n^{n/2}|\det P|,$$

*where $C_n = \gamma_n/n$ and $\gamma_n$ is Hermite's constant. Hence if $L_P$ cuts infinitely many sets $K(u)$, then this inequality has infinitely many solutions in $u \in \mathbf{Z}^n$.*

A component of the boundary of a set $R(u)$ of codimension one will be called a *face* of $R(u)$. We shall say that sets $R(u)$ and $R(u')$ are *neighbors* if they have a common face, and if the sets $R(u)$ and $R(u')$ are neighbors then the convergents $u$ and $u'$ are *neighbors*. The following lemma can be used to find the faces of $R = L_P \cap K(w) \neq \varnothing$ (see e.g. Example 4 below).

**Lemma 8.** *Let $L_P$ be the axis of $g \in G$. Assume that $R = L_P \cap K(w) \neq \varnothing$. Let $R_i = L_P \cap K(u_i)$, $i = 1, 2, ...$, be the neighbors of $R$, so that $R$ and $R_i$ have a common face $\phi_i$. Then $\phi_i \subset L^+(u_i)$.*

*Proof.* Assume that $K(w)$ and $K(gw)$ have a common face and that $X \in \overline{K}(w) \cap \overline{K}(gw)$. Then $X[g] \in \overline{K}(w)$. Hence $X[w] = X[gw] = 1$ and $X \in L^+(gw)$. Thus, the common face of $K(w)$ and $K(gw)$ lies in $L^+(gw)$. □

Algorithm II below can be used to enumerate all the sets $R(u)$ which form the tessellation of $L_P$. Our ability to apply this algorithm is based on the assumption that one can find all the faces of $R = L_P \cap K(w)$ for any $L_P$. Then all the faces of any $R_i = L_P \cap K(u_i) \neq \varnothing$, where $u_i = g_iw$ and $g_i \in \Gamma$, can be found since $R_i[g_i] = L_P[g_i] \cap K(w)$. It is clear that $g_i$ is not unique. Algorithm II will make the choice of $g_i$ more specific.

Let $D$ be a fundamental domain of $\Gamma$. The *floor* of $D$ consists of the faces of $D$ which do not contain the point $W$. Let $\mathcal{F}_D = \{\phi_1, ..., \phi_m\}$ be the set of faces in the floor of $D$. Let $\mathcal{S}_D = \{S_1, ..., S_m\}$, $S_i \in \Gamma$, where the face $\phi_i \subset L^+(S_i^{-1})$. When $n = 2$ or 3, the floor of $D$ consists of only one face, and one can choose $S_1$ to be the reflection with respect to this face (see [33], p. 1310, and Example 1).

As mentioned above, the tessellation of $L_P$ is invariant with respect to the action of $\Gamma$. Hence, we can assume that $L_P \cap K(w) \neq \varnothing$. Indeed, if $L_P \cap K(w) = \varnothing$, take a point $X \in L_P$ and find $h \in \Gamma$ such that $X[h] \in K(w)$. (Any of the reduction algorithms (see e.g. [8] for references) can be used to find such an $h$.) Then $L_P[h] \cap K(w) \neq \varnothing$.

Denote by $V_L$ the set $\{R(u)\}$, where the sets $R(u)$ form the tessellation of $L = L_P$. There is a unique graph $\overline{G}_L = (V_L, \overline{E}_L)$ associated with $L$ whose set of vertices is $V_L$, and there is an edge $(R, R') \in \overline{E}_L$ if and only if $R$,$R' \in V_L$ are neighbors.

**Algorithm II.** This algorithm finds a spanning tree $G_L = (V_L, E_L)$ of the graph $\overline{G}_L$. An edge $(R, R') \in E_L$ is labeled by $T \in \Gamma$ if $R = L \cap K[g]$ and $R' = L \cap K[Tg]$. If the dimension of $L$ is one, then $\overline{G}_L = G_L$.

**Input**. A simplex $L \subset \mathcal{P}$ with vertices at $A_i \in \mathcal{B}$, where $A_i = a_i a_i^T$ for $i = 1, ..., s$, $A_{s+i} = \alpha_{s+i}\alpha_{s+i}^T + \beta_{s+i}\beta_{s+i}^T$ for $i = 1, ..., t$, $a_i, \alpha_{s+i}, \beta_{s+i} \in \mathbf{R}^n$, and $\det(a_1, ..., a_s, \beta_{s+1}, \alpha_{s+1}, ..., \beta_{s+t}, \alpha_{s+t}) \neq 0$.

$R_0 = L \cap K(w) \neq \varnothing$.

**Output**. A tree $G_L = (V_L, E_L)$ where $V_L = \{R(u)\}$, where sets $R(u)$ form the tessellation of $L$.

Denote by $(\mathcal{V}, \mathcal{E})$ a subtree of $G_L$ where $\mathcal{V} \subset V_L$ and $\mathcal{E} \subset E_L$ are current sets of vertices and edges of $G_L$ found.

Let $\mathcal{L}$ be an ordered list of leaves of the subtree $(\mathcal{V}, \mathcal{E})$ of $G_L$.

Let $R \in \mathcal{L}$. Let $\{R_k,\, k = 1, 2, ...\}$ be the set of neighbors of $R$ such that $R_k \notin \mathcal{V}$. Denote by $\mathcal{N}(R)$ an ordered set of faces $\psi_k = \overline{R} \cap \overline{R}_k$.

The root of $G_L$ is $R_0 = R(w)$.

$\mathcal{V} = \mathcal{L} = \{R_0\}$ and $\mathcal{E} = \{\varnothing\}$.

**Step 1**. Let $\psi_k \in \mathcal{N}(R_0)$. Find $U_k \in \Gamma_\infty$ such that $\psi_k \subset \phi_k[U_k]$ for some $\phi_k \in \mathcal{F}_D$. Let $L'_k = L[U_k^{-1}]$.

Then the face $\psi'_k = \psi_k[U_k^{-1}]$ of $R' = L'_k \cap K(w)$ lies in $\phi_k$.

**Step 2**. Let $\psi'_k \subset \phi_k \subset L^+(S_k^{-1})$, where $S_k \in \mathcal{S}_D$, be as in step 1. Denote $T_k = S_k U_k$.

(Then $L_k = L'_k[S_k^{-1}] = L[T_k^{-1}]$ cuts $K(w)$, and $L = L_k[T_k]$.)

Add $R_k = L \cap K[T_k]$ to $\mathcal{V}$ and $\mathcal{L}$, and add the edge $(R_0, R_k)$ labeled by $T_k$ to $\mathcal{E}$.

**Step 3.** If $\mathcal{N}(R_0) \neq \varnothing$, then go to step 1. Otherwise, remove $R_0$ from $\mathcal{L}$.

(When $R_0$ is removed from $\mathcal{L}$, $\mathcal{L}$ consists of all the neighbors of $R_0$ and $\mathcal{V} = L \cup R_0$.)

**Step 4**. Let $R \in \mathcal{L}$. Let $(R_0, R_1, ..., R_{i-1}, R)$ be the path in $G_L$ from $R_0$ to $R$, whose edges $(R_{s-1}, R_s)$ are labeled by $T_s \in \Gamma$, $s = 1, ..., i$, so that

$$R_s = L \cap K[T_s \ldots T_1]$$

and $R = L \cap K[g]$, where $g = T_i \ldots T_1$.

(Then $R[g^{-1}] = L[g^{-1}] \cap K(w) \neq \varnothing$. If $L[g^{-1}] = L$, then $R[g^{-1}] = R_0$ and $g \in \Gamma_L$.)

**Step 5**. Let $\psi_k \in \mathcal{N}(R)$. Find $U_k \in \Gamma_\infty$ such that $\psi_k[g^{-1}] \subset \phi_k[U_k]$ for some $\phi_k \in \mathcal{F}_D$. Let $L'_k = L[g^{-1}U_k^{-1}]$.

Then the face $\psi'_k = \psi_k[g^{-1}U_k^{-1}]$ of $R' = L'_k \cap K(w)$ lies in $\phi_k$.

**Step 6**. Let $\psi'_k \subset \phi_k \subset L^+(S_k^{-1})$, where $S_k \in \mathcal{S}_D$, be as in step 6. Denote $T_k = S_k U_k$.

(Then $L_k = L'_k[S_k^{-1}] = L[g^{-1}T_k^{-1}]$ cuts $K(w)$, and $L = L_k[T_k g]$.)

Add $R_k = L \cap K[T_k g]$ to $\mathcal{V}$ and $\mathcal{L}$, and add the edge $(R, R_k)$ labeled by $T_k$ to $\mathcal{E}$.

**Step 7.** If $\mathcal{N}(R) \neq \varnothing$, then go to step 5. Otherwise, remove $R$ from $\mathcal{L}$ and go to step 4.

(When $R$ is removed from $\mathcal{L}$, all the neighbors of $R$ have been added to $\mathcal{V}$.)

*Remarks.* 1. By (5), $R = L \cap K[g] = R(g^{-1}w)$, and the convergent associated with the set $R$ is $u = g^{-1}w$, where $g = T_i \ldots T_1$. Thus, Algorithm II can be used to find all the convergents of $L$.

2. Assume that the sets $R_i$ form the tessellation of the axis $L_P$ of an irreducible $g \in \Gamma$. Then the number of non-congruent sets $R_i$ is finite and the tessellation of $L_P$ is periodic. Let $R_1 \cup \ldots \cup R_p = D_L$ be a fundamental domain of $\Gamma_L = \langle E_1, \ldots, E_{s+t-1} \rangle$ in $L_P$. Let $V'_L = \{u_1, \ldots, u_p\}$ be the set of all convergents of $L_P$ associated with the vertices $R_1, \ldots, R_p \in V_L$. Then the set of all convergents of $L_P$ is $\{gu : g \in \Gamma_L, u \in V'_L\}$. In Example 1 above, the set of convergents of $L_P$ is $\{E^n w, n \in \mathbf{Z}\}$. In Example 4 below, $p = 4$. In all other examples, $p = 1$, i.e., it will be shown in each of these cases that $R(w)$ is a fundamental domain of $\Gamma_L$.

3. If vertices $R_k$ on level $i$ of the tree $G_L$ have been added to $\mathcal{V}$ in step 6, then all the vertices of $G_L$ on levels $< i$ belong to $\mathcal{V}$. If $L$ is the axis of an irreducible $g \in \Gamma$, then in a finite number of steps $\mathcal{V}$ will contain a complete set of non-congruent sets $R(u)$. Thus, Algorithm II finds a fundamental domain and a set of generators of the group $\Gamma_L$ in a finite number of steps.

4. For the axis $L_P$ defined by (4), it can be shown that $\langle u, a_1 \rangle \to 0$ as $R(u) \to A_1 = a_1 a_1^T$ (see [33], Lemma 9).

## 4. Units in totally real fields

Let $(1, \omega_2, \ldots, \omega_n)$ be a $\mathbf{Z}$-basis of the ring of integers $\mathbf{Z}_F$ of a number field $F$ of degree $n$. Let $a_1 = (1, \omega_2, \ldots, \omega_n)^T$. Let $\gamma \in \mathbf{Z}_F$. Then $\gamma\omega_i = \sum m_{ij}\omega_j$ or $\gamma a_1 = M_\gamma a_1$, where $\omega_1 = 1$, $m_{ij} \in \mathbf{Z}$ and $M_\gamma = (m_{ij})$ is a square matrix of order $n$. As explained in Section 1, the map $\gamma \longmapsto M_\gamma$ is an isomorphism of the ring of integers $\mathbf{Z}_F$ and the commutative ring of $\mathbf{Z}$-integral square matrices of order $n$ with the common axis $L_P$ defined by (4), where $(s, t)$ is the signature of $F$. The norm of $\gamma$ equals $\det(M_\gamma)$, so that $\gamma$ is a unit in $\mathbf{Z}_F$ if and only if $M_\gamma \in \mathrm{GL}_n(\mathbf{Z})$. The group of units in $\mathbf{Z}_F$ is not isomorphic to $\mathrm{Stab}(L_P, \Gamma)$. On the one hand, $\Gamma = \mathrm{GL}_n(\mathbf{Z})/\{\pm 1\}$. On the other hand, the group $\mathrm{Stab}(L_P, \Gamma)$ is not necessarily commutative, since there may exist a non-trivial homomorphism $\mathrm{Gal}(F) \to \mathrm{Stab}(L_P, \Gamma)$. However, the torsion-free subgroup $\Gamma_L$ of $\mathrm{Stab}(L_P, \Gamma)$ is isomorphic to $\mathbf{Z}_F^\times/\mu_F$. Thus, the problem of finding a system of fundamental units of $F$ is equivalent to the problem of finding a set of generators of $\Gamma_L$. If $s > 0$, then the torsion group $\mu_F = \{\pm 1\}$, and $\Gamma_L$ is isomorphic to $\mathbf{Z}_F^\times/\{\pm 1\}$. Note that, by Lemma 5,

$$\mathrm{ht}(L_P) = \frac{1}{n}|d(F)|^{1/n},$$

where $d(F) = \det^2(P)$ is the *discriminant* of $F$.

A point $X = (x_{ij}) \in \mathcal{P}$ is said to be rational over the field $K$ if all $x_{ij} \in K$. A subset $S$ of $\mathcal{P}$ is rational over $K$ if the set of rational points of $S$ is dense in $S$. It is clear that the summit $q_m$ (see (6)) of $L_P$ is rational over some real subfield $F_L$ of the Galois closure of $F$. Let $\Gamma_L(\mathbf{Q})$ be the stabilizer of $L_P$ in $GL_n(\mathbf{Q})$. The $\Gamma_L(\mathbf{Q})$-orbit of $q_m$ is dense in $L_P$. Hence $L_P$ is rational over $F_L$. Assume that $\dim(L_P) = r$. Then any vertex of $R = L_P \cap K(w)$ is the intersection of $L_P$ with $r$ rational (over $\mathbf{Q}$) planes $L^+(g_i)$. Hence all the vertices of $R$ are rational over $F_L$. In this section,

$F$ is a totally real field, $\dim(L_P) = n-1$, and $F_L = \mathbf{Q}$. In Section 5, the signature of $F$ is (2,1).

Let $f(x) = x^n - c_{n-1}x^{n-1} - ... - c_1 x - c_0$ be an irreducible polynomial with integral coefficients. Let $f(\delta) = 0$. Let $F = \mathbf{Q}(\delta)$. Assume that $\mathbf{Z}_F$ has the *power* basis $\{1, \delta, ..., \delta^{n-1}\}$. Let $a_1 = (1, \delta, ..., \delta^{n-1})^T$. The integral matrix $C$ such that $Ca_1 = \delta a_1$ is said to be the *companion matrix* of $f(x)$. Let $a_i = (1, \delta_i, ..., \delta_i^{n-1})^T$, $i = 1, 2, ..., n$, where $\delta_1 = \delta$. An equation of the axis $L_P$ of $C^T$ is

$$q = \sum \mu_i A_i, \quad A_i = (a_{km}) = a_i a_i^T,\ \mu_i > 0,\ \mu_1 + ... + \mu_n = 1,$$

where $a_{km} = \delta_i^{k+m-2}$. Thus a point $q = (q_{km})$ of $L_P$ can be identified with the vector $[p_0, p_1, p_2, ..., p_{2n-2}]$, where $q_{km} = p_{k+m-2}$ and

$$p_i = \mu_1 \delta_1^i + ... + \mu_n \delta_n^i,\ i = 0, 1, ..., n-1.$$

Since $\delta_j^n = c_{n-1}\delta_j^{n-1} + ... + c_1\delta_j + c_0$,

(7)
$$p_i = c_{n-1}p_{i-1} + c_{n-2}p_{i-2} + ... + c_1 p_{i-n+1} + c_0 p_{i-n},\ i = n, n+1, ..., 2n-2.$$

Using $p_0$, ..., $p_{n-1}$ as parameters on $L_P$ instead of $\{\mu_i,\ i = 1, ..., n\}$, we obtain the following.

**Lemma 9.** *Assume that the ring of integers $\mathbf{Z}_F$ of a totally real field $F$ has a power basis. A point $q = [p_0, p_1, p_2, ..., p_{2n-2}]$, defined as above, belongs to the plane spanned by $L_P$ if and only if (7) holds, in which case the point $q \in L_P$ is uniquely determined by its first row:*

$$q = [p_0, p_1, ..., p_{n-1}].$$

In Example 1 above, $I = [1, 0]$ and $I[E] = [1, t]$ in these coordinates. In the examples below, we shall use these coordinates for a point $q \in L_P$ whenever the ring of integers $\mathbf{Z}_F$ of a totally real field $F$ has a power basis.

**Example 2.** Here we consider the case of the simplest cubic fields (see [25]). These are the cyclic fields of discriminant $(t^2+3t+9)^2$. The field $F = \mathbf{Q}(\epsilon_1)$ is generated by a root $\epsilon_1$ of $f(x) = x^3 - tx^2 - (t+3)x - 1$. Assume that $t^2+3t+9$ is square-free. Then $\{1, \epsilon_1, \epsilon_1^2\}$ is a basis of $\mathbf{Z}_F$, units $\epsilon_1$ and $\epsilon_2 = -1/(1+\epsilon_1)$ both are the roots of this polynomial, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon_1, \epsilon_2 \rangle$ (see [34]). For $t = -1$, 0, and 1, the discriminants of $F$ are $7^2, 9^2$, and $13^2$ respectively. These fields were considered by Davenport [9], [10], [11] (see also [3]), and Swinnerton-Dyer [26].

Let $E^T$ be the companion matrix of $f(x)$ and let $E_1 = E + I$. Let $L_P$ be the axis of $E$. Then the torsion-free subgroup $\Gamma_L$ of the stabilizer of $L_P$ in $\Gamma$ is generated by $E$ and $E_1$. Let $E^T a_i = \epsilon_i a_i$ and $A_i = a_i a_i^T$, where $a_i = (1, \epsilon_i, \epsilon_i^2)$, $i = 1, 2, 3$. Then $q(\mu_1, \mu_2, \mu_3) = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3$, $\mu_i > 0$, $\mu_1 + \mu_2 + \mu_3 = 1$, is an equation of $L_P$.

Let $R = L_P \cap K(w)$. An edge of $R$ is the intersection of $L_P$ with some $L^+(g)$, $g \in \Gamma$, which contains a face of $K(w)$, and the vertices of $R$ are the points of intersection of $L_P$ with some faces of $K(w)$ of codimension two. Denote $E_2 = EE_1^{-1}$. Let $F_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_2)$, and let $G_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_1)$. Since $\mathbf{Z}_F$ has a power basis, Lemma 9 is applicable. In our case, $R$ is the hexagon with vertices at $F_1 = [1, 0, 1]$, $F_2 = F_1[E]$, $F_3 = F_1[E_2]$, $G_1 = [1, -1/2, 1]$, $G_2 = G_1[E_1]$, $G_3 = G_1[E]$ with $\det(F_i) = 1$, $\det(G_i) = (t^2+3t+9)/8$. The sides of $R$ are identified as follows: $E : F_1G_1 \to F_2G_3$;

$E_1 : F_3G_1 \to F_2G_2$; $E_2 : F_1G_2 \to F_3G_3$. Since the forms $F_i$ are integral, they are extremal.

Let $t = 3u + k$, where $|k| \leq 1$. Denote

$$h = \begin{bmatrix} 1 & 0 & h_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}, \qquad G_0 = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & k/2 \\ 0 & k/2 & g_{33} \end{bmatrix},$$

where $h_{13} = -1 - u$, $h_{23} = -2u$, and $g_{33} = (4\det(G_1) + k^2)/3$. Then $G_0 = G_1[h]$ is Minkowski reduced ([8], pp. 396–397). Hence the points $G_i$ are extremal.

Hence $R = L_P \cap K(w)$ is a fundamental domain of $\langle E, E_1\rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon + 1\rangle$.

Note that if $F_1 = q(\mu_1, \mu_2, \mu_3)$, then $F_2 = q(\mu_2, \mu_3, \mu_1)$ and $F_3 = q(\mu_3, \mu_1, \mu_2)$. The same relations hold for $G_1, G_2$, and $G_3$. Also, for the summit $q_m$ of $L_P$, we have

$$q_m = \frac{1}{3}\sum A_i = \frac{1}{3}\sum F_i = \frac{1}{3}\sum G_i.$$

**Example 3.** Let $t \geq 3$ be a positive integer. Let $f(x) = x^3 + (t-1)x^2 - tx - 1$. Let $f(\epsilon) = 0$. Assume that the discriminant of $f(x)$ is square-free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of $\mathbf{Z}_F$, where $F = \mathbf{Q}(\epsilon)$ is a totally real field. The field $F$ is non-Galois, and it is exceptional; that is, $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon - 1\rangle$ (see e.g. [14], [21]). Since $\mathbf{Z}_F$ has a power basis, Lemma 9 is applicable.

Let $E^T$ be the companion matrix of $f(x)$, $E_1 = E - I$, and $E_2 = EE_1^{-1}$. Let $L_P$ be the axis of $E$. Let $F_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_1)$, and let $G_1$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_2)$. Let $c = t^2 + 3t + 1$. The region $R = K(w) \cap L_P$ is the hexagon with vertices at $F_1 = [1, 1/2, 1]$, $F_2 = F_1[E]$, $F_3 = F_1[E_1]$, $G_1 = [1, -(2t+4)/c, 1]$, $G_2 = G_1[E]$, and $G_3 = G_1[E_2]$ with $\det(F_i) = (t^2 + 3t - 9)/8$ and $\det(G_i) \sim 1$ as $t \to \infty$. Let $h$ be as in Example 2 with $h_{13} = -u$, $h_{23} = -2 + 2u$. Then $F_1[h]$ is Minkowski reduced. Hence the points $F_i$, $i = 1, 2, 3$, are extremal. Denote

$$h = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & -(2t+4)/c & (2t+2)/c \\ -(2t+4)/c & 1 & -2/c \\ (2t+2)/c & -2/c & 1 \end{bmatrix}.$$

The point $G_0 = G_1[h]$ is Minkowski reduced ([8], pp. 396-397); hence the points $G_i$, $i = 0, 1, 2, 3$, are extremal. The sides of $R$ are identified as follows: $E : F_1G_1 \to G_2F_2$; $E_1 : F_1G_3 \to F_3G_2$; $E_2 : F_3G_1 \to F_2G_3$. Hence $R$ is a fundamental domain of $\langle E, E_1\rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon - 1\rangle$.

Note that now neither $(\sum F_i)/3$ nor $(\sum G_i)/3$ is the summit of $L_P$.

**Example 4.** Let $f(x) = x^3 + (t+2)x^2 + (2t-1)x - 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that the discriminant of $f(x)$ is square-free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of $\mathbf{Z}_F$, where $F = \mathbf{Q}(\epsilon)$ is a totally real field. The field $F$ is non-Galois, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon + 2\rangle$ (see [21]). When $t = 3$, the discriminant of the field is 148. This case was considered by Swinnerton-Dyer [26] (see also [23]).

Let $E^T$ be the companion matrix of $f(x)$, $E_1 = E + 2I$, and $E_2 = EE_1$. Let $L_P$ be the axis of $E$. Since $\mathbf{Z}_F$ has a power basis, Lemma 9 is applicable.

Let $d = 2t^2 - 5t + 1$. Denote

$$F_1 = [1, -1/2 + 1/(4t-6), 1], \quad F_2 = F_1[E], \quad F_3 = F_1[E_2],$$
$$G_1 = [1, -1, 2], \quad G_2 = G_1[E_2],$$
$$H_1 = [1, -3/2, 3], \quad H_2 = H_1[E_1],$$
$$K_1 = [1, (2-t)/d, 1], \quad K_2 = K_1[E],$$
$$M_1 = [1, -1/(2t-1), 1], \quad M_2 = M_1[E_1^{-1}], \quad M_3 = M_2[E_2]$$

with $\det(G_1) = 1$, $\det(H_1) = (3t+1)/8$, $\det(M_1) = (6t+1)(t-1)^2/(2t-1)^3$, and $\det(F_1)$,$\det(K_1) \sim 3/4$ as $t \to \infty$.

Denote $u_1 = (0\ t\ 1)^T$, $u_2 = (1\ 1\ 0)^T$, $u_3 = (t\ 1\ 0)^T$ and $h_1 = [u_1, e_1, e_2]$, $h_i = [u_i, e_1, e_3]$, $i = 2, 3$, where $[e_1, e_2, e_3]$ is the identity matrix. Then $F_1 = L_P \cap L^+(E) \cap L^+(E_2)$, $G_1 = L_P \cap L^+(E_2) \cap L^+(u_2)$, $H_1 = L_P \cap L^+(E_1) \cap L^+(u_2)$, $K_1 = L_P \cap L^+(E) \cap L^+(u_1)$ and $M_1 = L_P \cap L^+(E) \cap L^+(E_1^{-1})$.

The points $G_1$ and $H_1$ are extremal since they are integral. Let

$$h(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $F_1[h(2)]$ and $K_1[h(t)]$ are Minkowski reduced ([8], pp. 396-397). Hence $F_i$ and $K_i$ are also extremal. The points $M_i$, $i = 1, 2, 3$, are not extremal. But the points $M_i' = (2t-1)/(2t-2)M_i[h_i]$ are extremal with $\det(M_i') = (6t+1)/(8t-8)$. In particular, $M_1[h_1]$ is Minkowski reduced.

The fundamental domain of $\Gamma_L = \langle E, E_1 \rangle$ is the hexagon $D_L = R \cup R_1 \cup R_2 \cup R_3$ with vertices at $F_1$, $M_1$, $F_3$, $M_3$, $F_2$ and $M_2$. The region $R = L_P \cap K(w)$ is the 9-gon with vertices at $F_1$, $K_1$, $H_2$, $F_3$, $G_2$, $K_2$, $F_2$, $H_1$ and $G_1$. Regions $R_1$, $R_2$ and $R_3$ are triangles. The vertices of $R_1$ are $M_1$, $K_1$ and $H_2$, of $R_2$ are $M_2$, $H_1$ and $G_1$, and of $R_3$ are $M_3$, $G_2$ and $K_2$. The intersection of the plane $L^+(u_1)$ with $D_L$ is the interval $K_1H_2$ which is the common boundary of $R$ and $R_1$. Similarly, $L^+(u_2) \cap D_L = G_1H_1 = R \cap R_2$ and $L^+(u_3) \cap D_L = G_2K_2 = R \cap R_3$. Note that $R_i[h_i] = L_P[h_i] \cap K(w)$, $i = 1, 2, 3$.

The sides of $D_L$ are identified as follows: $E : F_1M_1 \to F_2M_3$; $E_1 : F_2M_2 \to F_3M_1$; $E_2 : F_1M_2 \to F_3M_3$. Hence $D_L$ is a fundamental domain of $\langle E, E_1 \rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon + 2 \rangle$.

The set of convergents of $L_P$ is $\{gu, g \in \Gamma_L, u \in V_L'\}$, where $V_L' = \{w, u_1, u_2, u_3\}$.

*Remark.* Six sides of $R$ are identified as follows: $E : F_1K_1 \to F_2K_2$; $E_1 : F_2H_1 \to F_3H_2$; $E_2 : F_1G_1 \to F_3G_2$.

**Example 5.** Assume that $t > 3$ is a positive integer which is not divisible by 3. Then $4t^2 + 9$ is a square-free integer. Let $f(x) = x^4 + tx^3 - x^2 - tx + 1 = x(x^2 - 1)(x + t) + 1$. Let $f(\epsilon) = 0$. The discriminant of $f(x)$ is $(4t^2 + 9)(t^2 - 4)^2$. Let $\eta = \epsilon - \epsilon^{-1}$. Since

$$x^{-2}f(x) = (x - x^{-1})^2 + t(x - x^{-1}) + 1,$$

it follows that $\eta^2 + t\eta + 1 = 0$, $\eta = (-t \pm \sqrt{t^2 - 4})/2$, and $K = \mathbf{Q}(\eta)$ is a quadratic subfield of the totally real quartic field $F = \mathbf{Q}(\epsilon)$. We have $\epsilon^2 - \eta\epsilon - 1 = 0$, and the roots of $f(x)$ are

$$\epsilon_{i,i+2} = \frac{\eta_i}{2} \pm \sqrt{\frac{\eta_i^2}{4} + 1}, \quad \eta_i = -\frac{t}{2} \pm \sqrt{\frac{t^2}{4} - 1}, \quad i = 1, 2.$$

Clearly, $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4 = -1$. Hence, $F_i = K(\sqrt{\eta_i^2+4})$ and the discriminant of $F_i$ is

$$D_F = D_K^2 N(D_{F/K}) = (t^2-4)^2 N(\eta_i^2+4) = (t^2-4)^2(4t^2+9)$$

provided $t^2-4$ is the discriminant of $K$, in which case $\mathbf{Z}_K^\times/\{\pm 1\} = \langle \eta_i \rangle$, where $\mathbf{Z}_K^\times$ is the unit group in the maximal order $\mathbf{Z}_K$ of $K$. Since $N(D_{F/K}) = 4t^2+9$ is square-free, $\{1, \epsilon\}$ is a basis of $\mathbf{Z}_{F/K}$ (see e.g. [7], p. 79). It follows that $\{1, \epsilon, \epsilon\eta, \eta\}$ or $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of $\mathbf{Z}_F$. The regulator obtained from the units $\epsilon_1$, $\epsilon_2$, and $\eta$ is of order $\log^3 t$ as $t \to \infty$. To show that the Galois group of $\mathbf{Q}(\epsilon)$ is $D_4$, it is enough to show that $F_i$ is not a normal field. But, if it is, then $\sqrt{\eta_1^2+4}\sqrt{\eta_2^2+4} = \sqrt{N(\eta_i^2+4)} = \sqrt{4t^2+9}$ belongs $F_i$ and therefore $D_F$ is divisible by $(4t^2+9)^2$, which is not the case since $t > 3$.

Let $E^T$ be the companion matrix of $f(x)$, $E_1 = E + I$, $E_2 = E - E^{-1}$, and $E_3 = E - I = EE_2E_1^{-1}$. Let $L_P$ be the axis of $E$. Since $\mathbf{Z}_F$ has a power basis, Lemma 9 is applicable. Now we shall show that the following 28 points are the vertices of $R = L_P \cap K(w)$.

The points $A = [1, 1/2, 1, -2/t]$ and $B = [1, -1/2, 1, -1/2 - 2/t]$ with $\det(B) = \det(A) \sim \frac{3}{8}t$ as $t \to \infty$ are the intersections of $L_P$ with $L^+(E)$, $L^+(E_2)$, $L^+(E_3)$ and with $L^+(E_1)$, $L^+(E)$, $L^+(EE_3)$ respectively. Denote $A_1 = A[E]$, $A_2 = A[E_2]$, $A_3 = A[E_3]$, and $B_1 = B[E_1]$, $B_2 = B[E]$, $B_3 = B[EE_2]$.

The points $C = [1, -1/2, 1, -t/2 + 1/t - 1/4]$ and $D = [1, 1/2, 1, -t/2 + 1/t + 1/4]$ with $\det(D) = \det(C) \sim \frac{7}{64}t^2$ as $t \to \infty$ are the intersections of $L_P$ with $L^+(E_1)$, $L^+(E_1E_2^{-1})$, $L^+(E)$ and with $L^+(E)$, $L^+(EE_1^{-1})$, $L^+(E_3)$ respectively. Denote $C_1 = C[E_1]$, $C_2 = C[E_1E_2^{-1}]$, $C_3 = C[E]$ and $D_1 = D[E]$, $D_2 = D[EE_1^{-1}]$, $D_3 = D[E_3]$.

Let $s = 1/(2t^2 - t - 2)$. The points $K = [1, -s, 1, 2/t - t]$ and $L = [1, s, 1, s + 2/t - t]$ with $\det(L) = \det(K) \sim 3/4$ as $t \to \infty$ are the intersections of $L_P$ with $L^+(E)$, $L^+(E_1E_2^{-1})$, $L^+(E_2^{-1})$ and $L^+(E_1)$, $L^+(E_1E_2^{-1})$, $L^+(E_1E^{-1})$ respectively. Denote $K_1 = K[E]$, $K_2 = K[E_1E_2^{-1}]$, $K_3 = K[E_2^{-1}]$ and $L_1 = L[E_1]$, $L_2 = L[E_1E_2^{-1}]$, $L_3 = L[E_1E^{-1}]$.

The point $M = [1, 0, 1, -2/t]$ with $\det(M) = (1 - 4/t^2)^2$ is the intersection of $L_P$ with $L^+(E)$, $L^+(E_2)$, and $L^+(EE_2)$. Denote $M_1 = M[E]$, $M_2 = M[E_2]$, $M_3 = M[EE_2]$. Note that the summit $q_m$ of the 3-flat $L_P$ is $q_m = \frac{1}{4}(M + M_1 + M_2 + M_3)$ with $\det(q_m) = D_F/4^4$.

To show that all the points enumerated above are extremal we shall use Minkowski reduction. (If for some $h \in \Gamma_\infty$ the point $X[h]$ is Minkowski reduced ([8], pp. 396-397), then $X$ is extremal). Below we shall indicate such an $h$ for one of the vertices in each of the $\Gamma_L$-orbits of vertices of $R$. The polytope $R$ is bounded by four octagons lying in $L^+(g^{\pm 1})$, $g = E$, $E_2$, eight pentagons lying in $L^+(g^{\pm 1})$, $g = E_1$, $E_3$, $EE_1^{-1}$, $E_1E_2^{-1}$, and four triangular faces lying in $L^+(g^{\pm 1})$, $g = EE_2$, $EE_2^{-1}$. It has 28 vertices, 42 edges and 16 faces. The projections of the boundary of $R$ into a plane which is 'perpendicular' to its octagonal faces are shown in Figure 1.

Let

$$U_A = \begin{bmatrix} 1 & 1 & -t & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & -1 & 1-t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

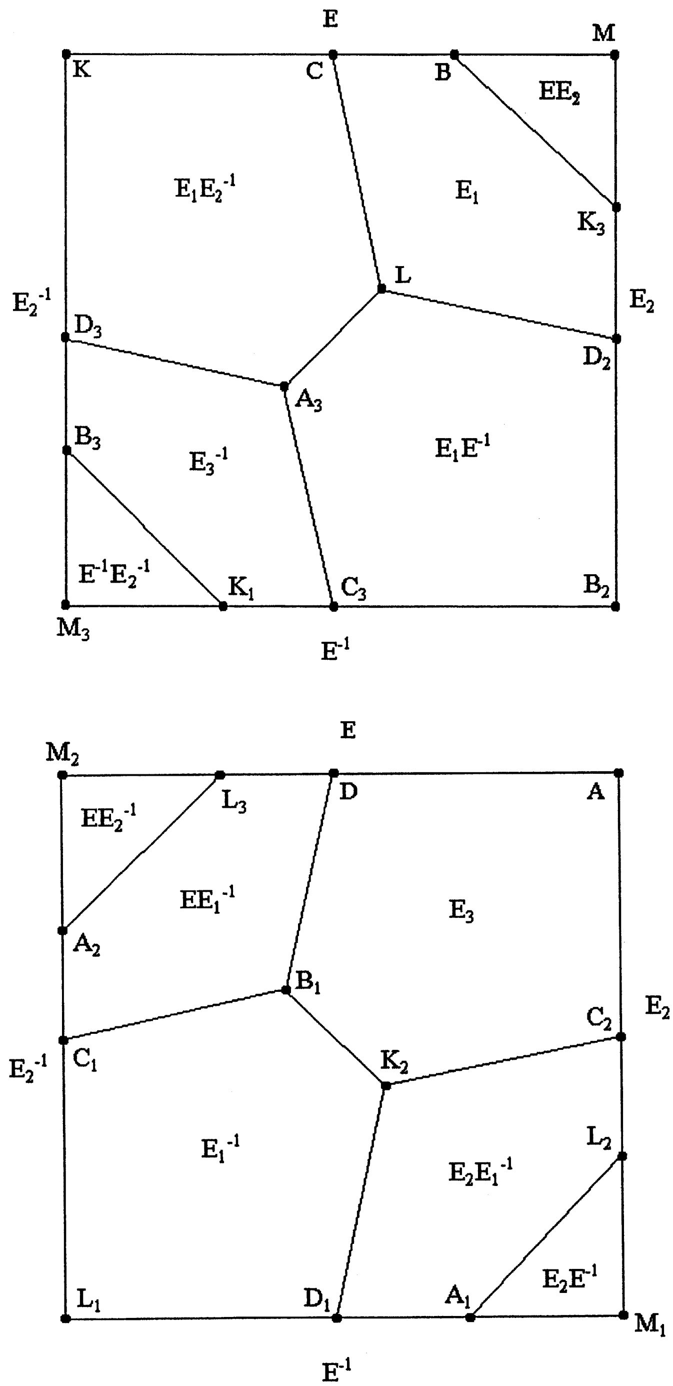
E
M
K
C
B
$EE_2$
$E_1E_2^{-1}$
$E_1$
$K_3$
L
$E_2^{-1}$
$E_2$
$D_3$
$D_2$
$A_3$
$B_3$
$E_3^{-1}$
$E_1E^{-1}$
$E^{-1}E_2^{-1}$
$K_1$
$C_3$
$B_2$
$M_3$
$E^{-1}$
E
$M_2$
$L_3$
D
A
$EE_2^{-1}$
$EE_1^{-1}$
$E_3$
$A_2$
$B_1$
$E_2$
$C_2$
$E_2^{-1}$
$C_1$
$K_2$
$L_2$
$E_1^{-1}$
$E_2E_1^{-1}$
$E_2E^{-1}$
$L_1$
$D_1$
$A_1$
$M_1$
$E^{-1}$

Figure 1.

Let $b = 1/2 - 2/t$, $c = t/2 + 1 - 4/t$, $d = -1/2 - 2/t$. Then

$$A^U = \begin{bmatrix} 1 & 1/2 & -2/t & -1/2 \\ 1/2 & 1 & 0 & -2/t \\ -2/t & 0 & 1 & -b \\ -1/2 & -2/t & -b & c \end{bmatrix}, \; B^U = \begin{bmatrix} 1 & -1/2 & 0 & b \\ -1/2 & 1 & -2/t & -1/2 \\ 0 & -2/t & 1 & 1/2 \\ b & -1/2 & 1/2 & c \end{bmatrix},$$

where $A^U = A[U_A]$ and $B^U = B[U_B]$, are Minkowski reduced. Hence $A$, $A_i$ and $B$, $B_i$, $1 = 1, 2, 3$, are extremal. Let

$$U_K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & t+1 & -t \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad U_L = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & t & -t \\ 0 & 0 & 1 & t-1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $a = 2t^3 - t^2 - 2t$, $b = 3t^2 - t - 4$, $c = -t^3 + t^2 + t$, $d = t - b$. Then

$$aK^U = \begin{bmatrix} a & -t & -d & -2a/t \\ -t & a & b & t^2 \\ -d & b & a & c \\ -2a/t & t^2 & c & a \end{bmatrix}, \; aL^U = \begin{bmatrix} a & t & t^2 & b \\ t & a & 2a/t & d \\ t^2 & 2a/t & a & c \\ b & d & c & a \end{bmatrix},$$

where $K^U = K[U_K]$ and $L^U = L[U_L]$, are Minkowski reduced. Hence $K$, $K_i$ and $L$, $L_i$, $1 = 1, 2, 3$, are extremal. For

$$U = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the matrix

$$tM[U] = \begin{bmatrix} t & 0 & 0 & -2 \\ 0 & t & -2 & 0 \\ 0 & -2 & t & 0 \\ -2 & 0 & 0 & t \end{bmatrix}$$

is Minkowski reduced. Thus, $M$, $M_i$, $i = 1, 2, 3$, are extremal.

Reduction of $C$ and $D$ is more difficult. Let $t = 22k + l$. Let

$$U_D = \begin{bmatrix} 1 & 0 & 1 & u_{14} \\ 0 & 1 & -t & u_{24} \\ 0 & 0 & t-1 & u_{34} \\ 0 & 0 & 1 & u_{44} \end{bmatrix},$$

where $u_{44} = -[t/11]$, $u_{34} = u_{44}(t-1) + 1$, $u_{14} = -9k - a$, while

$$u_{24} = 44k^2 + (2l + 15)k + b,$$

where $a = [l/3] + 1$, $b = [(2l + 1)/3]$ if $1 \le l \le 10$, and

$$u_{24} = 44k^2 + (2l - 7)k + b,$$

where $a = -[-l/3]$, $b = [(1 - l)/3]$ if $-11 \le l \le 0$. Then $D[U_D]$ is Minkowski reduced. For example, when $l = 0$,

$$D[U_D] = \begin{bmatrix} 1 & 1/2 & 1/4 + 1/t & 7/44 + 1/t \\ 1/2 & 1 & 1/4 - 1/t & 1/11 \\ 1/4 + 1/t & 1/4 - 1/t & 1 & 1/2 \\ 7/44 + 1/t & 1/11 & 1/2 & 7/44(t^2 - 4) \end{bmatrix}.$$

Let

$$U_C = \begin{bmatrix} 1 & 0 & -1 & u_{14} \\ 0 & 1 & t & u_{24} \\ 0 & 0 & t+1 & u_{34} \\ 0 & 0 & 1 & u_{44} \end{bmatrix},$$

where $u_{44} = -[t/11]$, $u_{34} = u_{44}(t+1)+1$, $u_{14} = 9k+a$, while

$$u_{24} = -44k^2 - (2l+7)k - b,$$

where $a = [l/3]$, $b = [(l+1)/3]$ if $0 \le l \le 11$, and

$$u_{24} = -44k^2 - (2l-15)k - b,$$

where $a = [l/3] - 1$, if $-10 \le l \le -1$, $l \ne -5$, and $a = [l/3] - 2$ if $l = -5$, and $b = [(1-2l)/3]$ if $-10 \le l \le -1$. Then $C[U_C]$ is Minkowski reduced. For example, when $l = 0$,

$$C[U_C] = \begin{bmatrix} 1 & -1/2 & 1/t - 1/4 & -1/11 \\ -1/2 & 1 & 1/t + 1/4 & 7/44 + 1/t \\ 1/t - 1/4 & 1/t + 1/4 & 1 & 1/2 \\ -1/11 & 7/44 + 1/t & 1/2 & 7/44(t^2-4) \end{bmatrix}.$$

Hence, $C$, $C_i$ and $D$, $D_i$, $1 = 1, 2, 3$, are extremal.

Thus, $R = L_P \cap K(w)$ is a fundamental domain of $\Gamma_L = \langle E, E_1, E_2 \rangle$, and therefore $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon + 1, \epsilon - 1/\epsilon \rangle = \langle \epsilon, \epsilon+1, \epsilon - 1 \rangle$.

**Example 6.** Let $f(x) = x^4 + tx^3 - x^2 - tx - 1 = x(x^2-1)(x+t) - 1$. Let $f(\epsilon) = 0$. The field $F = \mathbf{Q}(\epsilon)$ is totally real and $\mathrm{Gal}(F) = S_4$. Assume that the discriminant of $f(x)$ is square-free. Then $\mathbf{Z}_F$ has a power basis, and Lemma 9 is applicable.

Let $E^T$ be the companion matrix of $f(x)$, $E_1 = E - I$, $E_2 = E - E^{-1}$, and $E_3 = E + I = EE_2E_1^{-1}$. Let $L_P$ be the axis of $E$. We shall show that the region $R = L_P \cap K(w)$ is the same as in Example 5.

The point $A = [1, -1/2, 1, -1-2/t]$ with $\det(A) \sim \frac{3}{8}t$ as $t \to \infty$ is the intersection of $L_P$ with $L^+(E)$, $L^+(E_2)$ and $L^+(E_3)$. Denote $A_1 = A[E]$, $A_2 = A[E_2]$, $A_3 = A[E_3]$. Let $c = -1/2 + 2/t$,

$$U_A = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^U = \begin{bmatrix} 1 & -1/2 & -2/t & c \\ -1/2 & 1 & 0 & -c \\ -2/t & 0 & 1 & c + 2/t \\ c & -c & c+2/t & t/2 - 6c \end{bmatrix}.$$

Then $A[U_A] = A^U$ is Minkowski reduced. Hence $A$, $A_1$, $A_2$ and $A_3$ are extremal.

The point $B_3 = [1, 1/2, 1, 1/2]$ with $\det(B) = (6t-19)/16$ is the intersection of $L_P$ with $L^+(E^{-1}E_2^{-1})$, $L^+(E_2^{-1})$ and $L^+(E_3^{-1})$. Denote $B = B_3[E^{-1}E_2^{-1}]$, $B_1 = B_3[E_3^{-1}]$, $B_2 = B_3[E_2^{-1}]$. Since $B_3$ is an integral matrix, it is extremal.

The points $C = [1, 1/2, 1, -t/2 + 7/4 + 1/(2t-4)]$ and $D = [1, -1/2, 1, -t/2 - 7/4 + 1/(2t+4)]$ with $\det(C), \det(D) \sim \frac{7}{64}t^2$ as $t \to \infty$ are the intersections of $L_P$ with $L^+(E)$, $L^+(E_1)$, $L^+(E_1E_2^{-1})$ and with $L^+(E)$, $L^+(E_3)$, $L^+(EE_1^{-1})$ respectively. Denote $C_1 = C[E_1]$, $C_2 = C[E_1E_2^{-1}]$, $C_3 = C[E]$ and $D_1 = D[E]$, $D_2 = D[EE_1^{-1}]$, $D_3 = D[E_3]$. As in Example 5, reduction of $C$ and $D$ is divided into particular cases modulo 22. We consider only one case of such reduction. Let $t = 22k + 13$.

Let

$$U_C = \begin{bmatrix} 1 & 0 & -1 & -u_{14} \\ 0 & 1 & -t & -u_{24} \\ 0 & 0 & t-1 & u_{34} \\ 0 & 0 & 1 & u_{44} \end{bmatrix},$$

where $u_{44} = 2+2k$, $u_{34} = u_{44}(t-1)+1$, $u_{14} = 9k+6$, $u_{24} = 44k^2+55k+19$. Then

$$C[U_C] = \begin{bmatrix} 1 & 1/2 & -1/4+c & 1/22+c \\ 1/2 & 1 & -1/4-c & -1/4-1/22 \\ 1/4+c & 1/4-c & 1 & c \\ 1/22+c & -1/4-1/22 & c & 7/44t^2+3/11t-31/22-c \end{bmatrix},$$

where $c = 1/(22+44s)$, is Minkowski reduced. Hence $C$ is extremal. Similarly, the other cases of reduction of $C$ and $D$ can be considered.

Denote by $K_3$ the intersection of $L_P$ with $L^+(E_3)$, $L^+(E_2)$ and $L^+(EE_2)$, and by $L_3$ the intersection of $L_P$ with $L^+(E)$, $L^+(EE_2^{-1})$ and $L^+(EE_1^{-1})$. Let $K = K_3[E_2]$, $K_1 = K[E]$, $K_2 = K[E_1E_2^{-1}]$ and $L = L_3[EE_1^{-1}]$, $L_1 = L[E_1]$, $L_2 = L[E_1E_2^{-1}]$. Let

$$U_K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & t-1 & t \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad U_L = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & t & t \\ 0 & 0 & 1 & t+1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $a_1 = t(t-2)(2t+3)$, $b_1 = (t-4)/a_1$, $c_1 = (3t^2-4)/a_1$, $d_1 = (t^2-3t-4)/a_2$, $a_2 = t(t+2)(2t-5)$, $b_2 = (t+4)/a_2$, $c_2 = (3t^2-3t-8)/a_2$, $d_2 = -(t^2+2t-8)/a_2$. Then

$$\begin{aligned} K[U_K] &= \begin{bmatrix} 1 & -2/t-b_1 & c_1 & 2/t-2b_1 \\ -2/t-b_1 & 1 & d_1 & tb_1 \\ c_1 & d_1 & 1 & 1/2-tb_1/2 \\ 2/t-2b_1 & tb_1 & 1/2-tb_1/2 & 1 \end{bmatrix}, \\ L_3[U_L] &= \begin{bmatrix} 1 & -2/t-b_2 & -tb_2 & c_2 \\ -2/t-b_2 & 1 & 2b_2 & d_2 \\ -tb_2 & 2b_2 & 1 & 1/2-tb_2/2 \\ c_2 & d_2 & 1/2-tb_2/2 & 1 \end{bmatrix} \end{aligned}$$

are Minkowski reduced. Hence $K$ and $L$ are extremal. Note that $\det(K) \sim 3/4$, $\det(L) \sim 3/4$ as $t \to \infty$.

The point $M = [1, 2/t, 1, 2/t]$ with $\det(M) = (1-4/t^2)^2 - 4/t^2$ is the intersection of $L_P$ with $L^+(E)$, $L^+(E_2)$ and $L^+(EE_2)$. Denote $M_1 = M[E]$, $M_2 = M[E_2]$, $M_3 = M[EE_2]$. Let

$$U_M = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_U = \begin{bmatrix} 1 & 2/t & 0 & -2/t \\ 2/t & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/t \\ -2/t & 0 & 2/t & 1 \end{bmatrix}.$$

Since $M[U_M] = M_U$ is Minkowski reduced, the points $M$, $M_1$, $M_2$, $M_3$ are extremal.

Thus, the polytope $R = L_P \cap K(w)$ is the same as in Example 5 (see Figure 1). It is a fundamental domain of $\Gamma_L = \langle E, E_1, E_2\rangle$. Hence $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon+1, \epsilon - 1/\epsilon\rangle = \langle \epsilon, \epsilon+1, \epsilon-1\rangle$.

## 5. Units in fields with signature (2,1)

Let $n = 4$. Let $g \in \Gamma$. Assume that the characteristic polynomial of $g$ is irreducible with signature $(2,1)$. Let $\epsilon_1, \epsilon_2$ be real and $\epsilon_3, \epsilon_4 = \overline{\epsilon}_3$ non-real complex eigenvalues of $g$. Let $ga_i = \epsilon_i a_i$. Assume that the field $F = \mathbf{Q}(\epsilon_1)$ is dihedral and $K$ is its real quadratic subfield. Let $\sigma$ be the non-trivial automorphism of $F/K$. Then $\sigma(\epsilon_i) = \epsilon_{i+1}$, $i = 1, 3$. Hence $\sigma(A_1) = A_2$ and $\sigma(A_3) = A_3$, where $A_1$, $A_2$, and $A_3$ are vertices of $L_P$, the axis of $g$. Thus, the entries of $A_1 + A_2$ and of $A_3$ lie in $K$, and $q_m = (A_1 + A_2 + 2A_3)/3$ is rational over $K$. Hence $L_P$ and the vertices of $R = L_P \cap K(w)$ are rational over $K$. It follows that $F_L = K$ in this case. In Examples 7 and 9 below, the signature of the field $F$ is (2,1), $F$ has a real quadratic subfield $K$, and $F_L = K$. In Example 8, the degree of $F_L$ is six.

**Example 7.** Let $f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2+1)(x+t)+1$. Let $f(\epsilon) = 0$, where $\epsilon \in \mathbf{R}$. The discriminant of $f(x)$ is $-(4t^2-9)(t^2+4)^2$. Let $\eta = \epsilon + \epsilon^{-1}$. Since

$$x^{-2}f(x) = (x + x^{-1})^2 + t(x + x^{-1}) - 1,$$

it follows that $\eta^2 + t\eta - 1 = 0$, $\eta = (-t \pm \sqrt{t^2+4})/2$, and $K = \mathbf{Q}(\sqrt{d})$, $d = t^2 + 4$, is a quadratic subfield of the dihedral quartic field $F = \mathbf{Q}(\epsilon)$ with signature $(2,1)$. We have $\psi(\epsilon) = \epsilon^2 - \eta\epsilon + 1 = 0$, and the roots of $f(x)$ are

$$\epsilon_{i,i+2} = \frac{1}{2}(\eta_i \pm \sqrt{\eta_i^2 - 4}), \quad \eta_i = \frac{1}{2}(-t \mp \sqrt{t^2+4}), \quad i = 1, 2.$$

Hence $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4 = -1$. If $N(\eta^2 - 4) = 4t^2 - 9$, the norm of the discriminant of $\psi(x)$, is square-free, then $\{1, \epsilon\}$ is a basis of $\mathbf{Z}_{F/K}$. Hence $\{1, \epsilon, \eta\epsilon, \eta\}$ or $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of $\mathbf{Z}_F$ provided $\eta$ is a fundamental unit of $K$.

Let $E^T$ be the companion matrix of $f(x)$. Then $E^T a_i = \epsilon_i a_i$, where $a_i = (1, \epsilon_i, \epsilon_i^2, \epsilon_i^3)$. Denote $E_1 = E + E^{-1}$. Let $M$ be the intersection of $L_P$, $L^+(E)$, and $L^+(E_1)$. Then

$$M[x] = \mu_1(x, a_1)^2 + \mu_3(x, a_3)^2 + \mu_2|(x, a_2)|^2,$$

where $\mu_{1,3} = (1 + (t \pm (2t\sqrt{d} + 2t^2 - 12)^{1/2})/\sqrt{d})/4$, $\mu_2 = (1 - t/\sqrt{d})/2$, and $\det(M) = (4t^2 - 9)(t^2 + 2 - t\sqrt{d})/2$.

Let $b = -2t(\eta\sqrt{d})^{-1} - 3(2\sqrt{d})^{-1}$, $c = -2t(\eta\sqrt{d})^{-1} + 3(2\sqrt{d})^{-1}$, $e = 5(4\eta\sqrt{d})^{-1}$ and

$$h = \begin{bmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 1 & b & e & 0 \\ b & 1 & 0 & -e \\ e & 0 & 1 & c \\ 0 & -e & c & 1 \end{bmatrix}.$$

Then $M_0 = M[h]$ is Minkowski reduced (see e.g. [8], p. 397). Hence the points $M$, $M[E]$, $M[E_1]$, and $M[EE_1]$ are extremal, and they are the vertices of the 'square' $R = L_P \cap K(w)$, which is a fundamental domain of $\langle E, E_1\rangle$. Thus $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \eta\rangle$.

Note that $q_m = (M + M[E] + M[E_1] + M[EE_1])/4$ is the summit of $L_P$.

**Example 8** (cf. [20])**.** Let $t \geq 4$ and $f_t(x) = f(x) = x^4 + tx^3 + x^2 + tx - 1 = x(x^2+1)(x+t) - 1$. Note that $f_{-t}(-x) = f_t(x)$. Let $\epsilon$ and $\epsilon_1$ be the real and $\epsilon_2, \overline{\epsilon_2}$ non-real complex roots of $f(x) = 0$. The signature of $F = \mathbf{Q}(\epsilon)$ is $(2,1)$, and $\mathrm{Gal}(F) = S_4$. Assume that the discriminant of $F$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of the maximal order $\mathbf{Z}_F$ of $F$.

Let $E^T$ be the companion matrix of $f(x)$. Let $E_1 = E^2(E+tI)$. Let $L_P$ be the axis of $E$. It can be identified with the set

$$F(\nu_1,\nu_2,\nu_3) = \nu_1 a^T a + \nu_2 a_1^T a_1 + \nu_3(a_{2R}^T a_{2R} + a_{2I}^T a_{2I}),$$

where $\nu_k > 0$, $\nu_1+\nu_2+\nu_3 = 1$. Here $a = (1,\epsilon,\epsilon^2,\epsilon^3)$, $a_i = (1,\epsilon_i,\epsilon_i^2,\epsilon_i^3)$, $i = 1,2$, are the eigenvectors of $E$ corresponding to its eigenvalues $\epsilon,\epsilon_1,\epsilon_2$ respectively. Let $A$ and $B$ be the intersections of $L_P$ with $L^+(E)$, $L^+(E_1)$ and with $L^+(E^{-1})$, $L^+(E_1^{-1})$ respectively.

Let $\alpha = |\epsilon_2|^2 = -1/(\epsilon\epsilon_1)$. Then $\alpha \to 1$ as $t \to \infty$. Over $F_L = \mathbf{Q}(\alpha)$ we have $f(x) = f_1(x)f_2(x)$, where

$$\begin{aligned} f_1(x) &= x^2 + t\frac{\alpha+1}{\alpha^2+1}x - \frac{1}{\alpha}, \\ f_2(x) &= x^2 + t\frac{\alpha^2-\alpha}{\alpha^2+1}x + \alpha, \end{aligned}$$

so that $f_i(\epsilon_i) = 0$, $i = 1,2$. The degree of $F_L$ is 6, since $g(\alpha) = 0$, where $g(x) = x^6 - x^5 + (t^2+1)x^4 - 2x^3 - (t^2+1)x - x - 1$.

Let $d_1$ be the discriminant of $f_1(x)$. Let $\nu_{1,2} = \mu_1 \pm \mu_2\sqrt{d_1}$, $\mu_3 = \nu_3$, $a^T a = M_1 + M_2\sqrt{d_1}$, $a_1^T a_1 = M_1 - M_2\sqrt{d_1}$. Then $L_P$ can be identified with the set $\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3$, where the entries of $M_k$, $k = 1,2,3$, belong to $F_L$. Thus, the summit $q_m$ of $L_P$, $L_P$ itself and all the vertices of $R = L_P \cap K(w)$ are rational over $F_L$.

Let $\Delta_A$ be the determinant of the system of three linear equations

$$\begin{aligned} 2\mu_1 + \mu_3 &= 1, \\ (\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3)[wE] &= 1, \\ (\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3)[wE_1] &= 1, \end{aligned}$$

with respect to $\mu_1,\mu_2,\mu_3$. Then

$$\begin{aligned} D_A &= \Delta_A(\alpha^2+1)\alpha/(4t) \\ &= \alpha^5 + 2\alpha^4 + (2t^2-1)\alpha^3 + (t^2-2)(\alpha^2-\alpha) - 2. \end{aligned}$$

Let $\Delta_B$ be the determinant of the system of three similar linear equations where $E$ and $E_1$ are replaced by $E$ and $EE_1^{-1}$. Then

$$\begin{aligned} D_B &= \Delta_B(\alpha^2+1)/(4t) \\ &= -(\alpha^3 + (t^2+5)\alpha + t^2 + 2). \end{aligned}$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & t \\ 0 & 0 & t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and let $A_U = A[U_A] = (a_{ij})$ and $B_U = B[U_B] = (b_{ij})$. Then $a_{11} = a_{22} = a_{33} = 1$,

$$\begin{aligned}
2tD_A a_{12} &= (1-\alpha)(1+\alpha^2-\alpha^3)t^2 - 4\alpha(2-\alpha+2\alpha^2-\alpha^3),\\
2tD_A a_{13} &= (\alpha-\alpha^3)t^4 + (1+5\alpha-7\alpha^2+4\alpha^3+2\alpha^4-\alpha^5)t^2 - 6 - 14\alpha^2\\
&\quad +6\alpha^3 - 8\alpha^4 + 6\alpha^5,\\
2D_A a_{14} &= (1-\alpha)(1+\alpha-4\alpha^2)t^2 - 6 - 3\alpha - 13\alpha^2 + 2\alpha^3 - 7\alpha^4 + 5\alpha^5,\\
2D_A a_{23} &= (1-\alpha)(\alpha+6\alpha^2+\alpha^3)t^2 + 11 - 3\alpha + 18\alpha^2 - 9\alpha^3 + 7\alpha^4 - 6\alpha^5,\\
2tD_A a_{24} &= (1+2\alpha-2\alpha^2+7\alpha^3+\alpha^4-\alpha^5)t^2 - 6 + 8\alpha - 18\alpha^2 + 14\alpha^3\\
&\quad -12\alpha^4 + 6\alpha^5,\\
2tD_A a_{34} &= 4(\alpha^2-\alpha^3)t^4 + (9-9\alpha+23\alpha^2-14\alpha^3+5\alpha^4-4\alpha^5)t^2\\
&\quad +12 - 8\alpha + 32\alpha^2 - 20\alpha^3 + 20\alpha^4 - 12\alpha^5,\\
D_A a_{44} &= (2\alpha+3\alpha^2-3\alpha^3)t^2 + 8 + 12\alpha^2 + \alpha^3 + 2\alpha^4 - 3\alpha^5,
\end{aligned}$$

and $b_{11} = b_{22} = b_{33} = 1$,

$$\begin{aligned}
2tD_B b_{12} &= 4(\alpha^3-\alpha)t^4 + (-3-9\alpha-12\alpha^2+10\alpha^3-4\alpha^4+4\alpha^5)t^2\\
&\quad -4 + 2\alpha - 14\alpha^2 + 8\alpha^3 - 10\alpha^4 + 6\alpha^5,\\
2D_B b_{13} &= (\alpha^2-1)(2-3\alpha)t^2 - 4 - 11\alpha + 11\alpha^2 - 4\alpha^3 + 5\alpha^4 - 3\alpha^5,\\
2tD_B b_{14} &= (\alpha^2-1)(\alpha+2)t^4 + (-7-12\alpha-2\alpha^2-\alpha^3+\alpha^4+\alpha^5)t^2\\
&\quad +4\alpha(1-\alpha+\alpha^2-\alpha^3),\\
2tD_B b_{23} &= 2(\alpha^3-\alpha)t^4 + (5+4\alpha-7\alpha^2-2\alpha^3-2\alpha^4+2\alpha^5)t^2\\
&\quad +8 + 4\alpha + 8\alpha^2 - 4\alpha^5,\\
2D_B b_{24} &= 2(\alpha^3-\alpha)t^4 + (1-8\alpha-7\alpha^2+6\alpha^3-2\alpha^4+2\alpha^5)t^2\\
&\quad -3 - 13\alpha - 8\alpha^2 + 5\alpha^3 - 7\alpha^4 + 4\alpha^5,\\
2tD_B b_{34} &= 2(\alpha-\alpha^3)t^4 + (7+7\alpha-4\alpha^2-10\alpha^3+2\alpha^4-2\alpha^5)t^2\\
&\quad +10 + 14\alpha + 8\alpha^2 + 6\alpha^3 - 2\alpha^4 - 8\alpha^5,\\
D_B b_{44} &= (2-3\alpha-5\alpha^2)t^2 - 4 - 5\alpha - 4\alpha^2 + 7\alpha^3 - 6\alpha^4.
\end{aligned}$$

Let $c = (\alpha^2+1)^2$. The identity

$$t^2(\alpha^3-\alpha) = c - \left(\frac{c}{\alpha t}\right)^2 + \frac{c^3}{\alpha^2 t^2(\alpha^2 t^2 + c)}$$

implies $a_{ij} \to 0$ and $b_{ij} \to 0$ as $t \to \infty$ for $i \neq j$. It follows that $\det(A) \sim 1$ and $\det(B) \sim 3$ as $t \to \infty$. Thus, $A_U$ and $B_U$ are Minkowski reduced. Hence the points $A$, $A[E]$, $A[E_1]$, $B$, $B[E^{-1}]$, and $B[E_1^{-1}]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$. Thus, $R$ is a fundamental domain of $\langle E, E_1\rangle$, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \epsilon + t\rangle$.

**Example 9.** Let

$$f(x) = x^4 + stx^3 + (t-\alpha s^2)x^2 + s(t^2+2\alpha)x - \alpha = x(x^2+t)(x+st) - \alpha(sx-1)^2,$$

where $t \geq s$ are positive integers and $\alpha = \pm 1$. Let $d = t^2 + 4\alpha$. It can be easily verified that

$$f(x) = (x^2 + s\eta x - \alpha/\eta)(x^2 - \alpha s x/\eta + \eta),$$

where $\eta = \frac{1}{2}(t+\sqrt{d})$ satisfies the equation $\eta^2 - t\eta - \alpha = 0$. The roots of $f(x) = 0$ are

$$\epsilon_{i,i+2} = \frac{1}{2}(-s\eta_i \pm \sqrt{s^2\eta_i^2 + 4\alpha/\eta_i}), \quad \eta_i = \frac{1}{2}(t \pm \sqrt{d}), \quad i = 1, 2,$$

where $\eta_1 = \eta$ and real $\epsilon_1$ and $\epsilon_3$ are the roots of $x^2 + s\eta x - \alpha/\eta = 0$. The discriminant of $f(x)$ is $\Delta(f) = -d^2(4s^2t^3 + 12\alpha s^2 t - s^4 + 16\alpha)(1+s^2t)^2$. Let $\epsilon = \epsilon_1$. Since $\eta = (\epsilon^2 + t)/(1 - s\epsilon)$, $K = \mathbf{Q}(\eta)$ is a quadratic subfield of the dihedral quartic field $F = \mathbf{Q}(\epsilon)$ with signature (2,1). Assume that $\eta$ is a fundamental unit of $K$. Then $\{1, \eta\}$ is a basis of $\mathbf{Z}_K$. Let $c = ts^2 + 1$. Denote $p(x) = (s + (\alpha t - s^2)x + \alpha x^3)/c$. Then $\epsilon\eta = p(\epsilon)$. Assume that $4s^2t^3 + 12\alpha s^2 t - s^4 + 16\alpha$ is square-free. Then $\{1, \epsilon, \epsilon^2, p(\epsilon)\}$ is a basis of $\mathbf{Z}_F$, and the discriminant of $F$ is $D_F = \Delta(f)/(ts^2+1)^2$.

Let $a_i = (1, \epsilon_i, \epsilon_i^2, p(\epsilon_i))$. Let $E_0$ be the companion matrix of $f$. Denote

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s/c & (\alpha t - s^2)/c & 0 & \alpha/c \end{bmatrix}.$$

Let $E^T = \tau E_0 \tau^{-1}$, $E_1 = p(E)$, $E_2 = E_1 E^{-1}$. Then $E^T a_i = \epsilon_i a_i$, and the axis $L_P$ of $E$ has equation $q[x] = \mu_1(x, a_1)^2 + \mu_3(x, a_3)^2 + \mu_2|(x, a_2)|^2$, $\mu_i > 0$, $\mu_1 + \mu_3 + \mu_2 = 1$.

First let $\alpha = 1$. Let $A$ and $B$ be the intersections of $L_P$ with $L^+(E)$, $L^+(E_1)$ and with $L^+(E)$, $L^+(E_2^{-1})$ respectively. Then $\det(A) = \det(B)$. Let $a_{ii} = b_{ii} = 1$, $i = 1, 2, 3$, $a_{44} = b_{44} = \sqrt{d} - 1 + \eta^{-2}$, $a_{23} = b_{13} = 0$, $a_{14} = b_{24} = \eta^{-1}$,

$$\begin{aligned}
a_{24} &= -b_{14} = \left(-\frac{s}{2\sqrt{d}} + \frac{t}{s\eta^2\sqrt{d}}\right)(1 - \eta^{-1}), \\
a_{34} &= -b_{34} = \left(\frac{s}{2\eta\sqrt{d}} + \frac{t}{s\eta\sqrt{d}}\right)(1 - \eta^{-1}), \\
a_{13} &= -b_{23} = \frac{1}{s\eta} - \frac{2}{s\eta\sqrt{d}} - \frac{s}{2\eta^2\sqrt{d}}, \\
a_{12} &= -b_{12} = \frac{1}{s\eta^2} - \frac{2}{s\eta^2\sqrt{d}} + \frac{s}{2\eta\sqrt{d}}.
\end{aligned}$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad U_B = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & s & 1 \end{bmatrix}.$$

Then $A_U = (a_{ij}) = A[U_A]$ and $B_U = (b_{ij}) = B[E_2^{-1}][U_B]$ are Minkowski reduced. Hence the points $A$, $A[E]$, $A[E_1]$, $B$, $B[E]$, $B[E_2^{-1}]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$.

Now let $\alpha = -1$. Let $A$ and $B$ be the intersections of $L_P$ with $L^+(E^{-1})$, $L^+(E_2)$ and with $L^+(E_1)$, $L^+(E_2)$ respectively. Then $\det(A) = \det(B)$. Note that if $A = q(\gamma_1, \gamma_2, \gamma_3)$, then $B = q(\gamma_3, \gamma_2, \gamma_1)$. Here $\gamma_2 = \eta$ and $\sigma(\gamma_1) = \gamma_3$, where $\sigma \in$Gal$(F/K)$.

Let $a_{ii} = b_{ii} = 1$, $i = 1,2,3$, $a_{44} = b_{44} = \sqrt{d} - 1 - 2\eta^{-1} + \eta^{-2} - 4/(t\eta)$, $a_{14} = -b_{24} = -(1-2/t)/\eta$, $a_{23} = b_{13} = -2/t$,

$$\begin{aligned} a_{12} &= -b_{23} = -\frac{s}{2t\eta} - \frac{1}{s\eta^2}, \\ a_{13} &= -b_{12} = \frac{s}{2\eta^2 t} + \frac{1}{s\eta}, \\ a_{24} &= b_{34} = \left(\frac{s}{2\sqrt{d}}\left(1 - \frac{2}{t}\right) - \frac{t-2}{s\eta^2\sqrt{d}}\right)(1+\eta^{-1}), \\ a_{34} &= b_{14} = \frac{2t - s^2}{2s\eta\sqrt{d}}(1+\eta^{-1})(1-2/t). \end{aligned}$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -s \end{bmatrix}, \qquad U_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t-1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -s & 1 \end{bmatrix}.$$

Then $A_U = (a_{ij}) = A[E^{-1}][U_1^T]$ and $B_U = (b_{ij}) = B[U]$ are Minkowski reduced. Hence the points $A$, $A[E^{-1}]$, $A[E_2]$, $B$, $B[E_1]$, $B[E_2]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$.

Thus, for any $\alpha$, $R$ is a fundamental domain of $\langle E, E_2\rangle$, and $\mathbf{Z}_F^\times/\{\pm 1\} = \langle \epsilon, \eta\rangle$.

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